

1 **GLOBAL-IN-TIME SOLUTIONS FOR QUASILINEAR**
2 **PARABOLIC PDEs WITH MIXED BOUNDARY CONDITIONS IN**
3 **THE BESSEL DUAL SCALE**

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ABSTRACT. We prove existence and uniqueness of global-in-time solutions in the $W_D^{-1,p}$ - $W_D^{1,p}$ -setting for abstract quasilinear parabolic PDEs with non-smooth data and mixed boundary conditions, including a nonlinear source term with at most linear growth. Subsequently, we use a bootstrapping argument to achieve improved regularity of these global-in-time solutions within the functional-analytic setting of the interpolation scale of Bessel-potential dual spaces $H_D^{\theta-1,p} = [W_D^{-1,p}, L^p]_\theta$ with $\theta \in [0, 1]$ for the abstract equation under suitable additional assumptions. This is done by means of new nonautonomous maximal parabolic regularity results for nonautonomous differential operators operators with Hölder-continuous coefficients on Bessel-potential spaces. The upper limit for θ is derived from the maximum degree of Hölder continuity for solutions to an elliptic mixed boundary value problem in L^p .

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- 1 **1. Introduction.** This work is concerned with global-in-time existence of solutions
 2 $u: (0, T) \times \Omega \rightarrow \mathbb{R}$ to quasilinear parabolic equations of type

$$\left. \begin{aligned} \partial_t u - \operatorname{div}(\xi(u)\mu\nabla u) + u &= \mathcal{F}_\Omega(u) && \text{on } (0, T) \times \Omega, \\ \nu_{\partial\Omega} \cdot \xi(u)\mu\nabla u + \alpha u &= \mathcal{F}_\Gamma(u) && \text{on } (0, T) \times \Gamma_N, \\ u &= 0 && \text{on } (0, T) \times \Gamma_D, \\ u(0) &= u_0 && \text{on } \Omega, \end{aligned} \right\} \quad (1)$$

3 and their regularity. We will interpret (1) as an abstract evolution equation in a scale
 4 X_θ of function spaces and work within a maximal parabolic regularity framework.
 5 The scale will be $X_\theta := [W_D^{-1,p}(\Omega), L^p(\Omega)]_\theta$ with $p > d$. (All objects and notions
 6 will be properly introduced below.) The defining feature of (1) is the coefficient
 7 $\xi(u)$ in the divergence operator. The problem further includes mixed boundary
 8 conditions on the disjoint boundary parts Γ_D and Γ_N with $\Gamma_D \cup \Gamma_N = \partial\Omega$ where
 9 we allow for inhomogeneous Robin/Neumann data. The setting for $\Omega \subset \mathbb{R}^d$, where
 10 $d \in \{2, 3\}$, will be that of a bounded weak Lipschitz domain compatible with the
 11 mixed boundary conditions (regular in the sense of Gröger). The nonlinear functions
 12 $\mathcal{F}_\Gamma$ and $\mathcal{F}_\Omega$ have to satisfy a “local” Lipschitz condition and we can afford up to
 13 linear growth; this is a classical assumption when aiming for global-in-time results.
 14 (We do not assume monotonicity for this work.) Finally, we note that there will
 15 be no further explicit smoothness assumption on the coefficient matrix μ , we only
 16 assume it to be bounded. We will however require that the weak divergence operator
 17 $-\nabla \cdot \mu \nabla + 1$ admits optimal elliptic regularity in $W_D^{-1,p}(\Omega)$.

18 Our main result is that if the data in the abstract formulation of (1) yields a
 19 well-defined problem in $X_\theta := [W_D^{-1,p}(\Omega), L^p(\Omega)]_\theta$ with $p > d$, then this problem
 20 admits a unique global-in-time solution $u \in W^{1,s}(0, T; \operatorname{Dom}_{X_\theta}(-\nabla \cdot \mu \nabla + 1)) \cap$
 21 $L^s(0, T; X_\theta)$ for suitably large s . This is true for all $\theta \in [0, \bar{\theta}]$ for some $\bar{\theta} \in [1 -$
 22 $d/p, 1]$; thus, we have global-in-time solutions for a whole scale of function spaces
 23 at our disposal which allows for a flexible treatment of a wide range of applications.
 24 The upper bound $\bar{\theta}$ depends on the degree of Hölder regularity admitted by the
 25 solutions to the elliptic problem with mixed boundary conditions associated to
 26 $-\nabla \cdot \mu \nabla + 1$ in $L^p(\Omega)$. The reasoning is based on nonautonomous maximal parabolic
 27 regularity with constant domains for $-\nabla \cdot \eta \mu \nabla$ in X_θ when the coefficient $\eta > 0$ is
 28 Hölder continuous in space of order $> \theta$. This is a second main result. The former
 29 property is used to bootstrap the unique global-in-time solution which exists for
 30 $X_0 = W_D^{-1,p}(\Omega)$; this was established in previous work ([43]) and is revisited and
 31 improved below.

32 *Context.* Quasilinear parabolic PDEs arise in several real-world applications,
 33 for instance in the immediate example of heat conduction when conductivity is
 34 temperature-dependent, but also in much more complicated problems such as semi-
 35 conductor physics [50] or liquid crystal growth [29], see also [3, Introduction] and
 36 the references there. We also mention the classical monography [35] for early work
 37 in this context. More recently, several contributions [9, 17, 25, 30, 31, 40, 41, 43]
 38 have addressed a rather rough geometric setting for such problems with respect
 39 to the domain, boundary conditions, and coefficient functions, similar to the one
 40 listed above. Such a nonsmooth geometric setting as described before is often mo-
 41 tivated by a realistic and appropriate model of industrial applications where one
 42 has e.g. nonsmooth workpieces made of different materials. The regularity concept
 43 of Gröger has proven to be a sort of umbrella framework in this regard, see [19,

23] and the references therein. The approach for (1) then goes via the concept of (nonautonomous) *maximal parabolic regularity*, which has turned out to be quite flexible and useful in this context because it allows to “modularize” the functional-analytic treatment of abstract evolution equations into several building blocks. We refer to the fundamental recent works of Amann [4] and Prüss and collaborators [34, 36, 47] in this regard. This is particularly useful in the nonsmooth geometric setting, because many convenient tools from e.g. elliptic regularity theory or (function space) interpolation theory are not available at all or at least require further justification, and a modularized ansatz allows to make use of new developments in every component. In fact, these insights have sparked quite some research interest in the respective directions and, indeed, there has been tremendous development here in the recent years, even for (much) more general geometric settings; we exemplary refer to [6, 7, 10, 19, 26, 42].

However, the question of *global-in-time* solutions to problems of type (1) remains ubiquitous, at least as far as it can be expected from the respective physical model. In the nonsmooth geometric setup, existence of global-in-time solutions for a less general instance of the abstract version of (1) was established for $X_0 = W_D^{-1,p}(\Omega)$ for $p > d$ in [43]; the solution is then in $W^{1,s}(0, T; W_D^{-1,p}(\Omega)) \cap L^s(0, T; W_D^{1,p}(\Omega))$ with $\frac{1}{s} < \frac{1}{2} - \frac{d}{2p}$. The authors rely on uniform Hölder estimates for nonautonomous linear parabolic equations established in the same paper and nonautonomous maximal parabolic regularity; we will revisit and improve upon the result in Section 3.1. The insight here was that the elliptic differential operator depends on the coefficient perturbation $\xi(u)$ in a well suited way in the topology of uniformly continuous functions on $\bar{Q}$ with $Q := (0, T) \times \Omega$, and this combines very well with the result on Hölder continuity on $\bar{Q}$, which is uniform in the respective coefficient functions, within a Schauder fixed point argument. Following and extending this train of thought, the authors of [9] were able to show that the equation arising from inserting the given global-in-time solution in the nonlinear functions, and re-interpreting as a linear nonautonomous evolution equation, is well-posed in the Bessel-potential spaces $X = H_D^{\zeta,p}(\Omega)$ where $\zeta < 1$, but close to 1. This is based on the observation that the Hölder continuity of the given global solution is in fact sufficient to verify nonautonomous maximal parabolic regularity for the differential operator in this space and allows to bootstrap the regularity for u , if the respective data in the problem is suitable. In this work, we take this procedure even further: Since the given global solution is now more regular, its degree of Hölder continuity, at least in space, has now also increased, and indeed, we are able to show that this leads to another bootstrap-improvement of regularity for u . This bootstrapping will work as far as the given data admits. We will explain this in a bit more detail below. Before, let us mention that there is another recent result on global-in-time existence in [12] to a problem similar to (1), but in a more regular setting, that is, for pure homogeneous Dirichlet boundary conditions and a $C^{1,1}$ -domain. We comment on the relation of our work to [12] e.g. in Remark 5.13 below.

Motivation. In addition to a contribution to the framework of analysis based on maximal regularity techniques for interesting real-world applications as mentioned before, the original motivation of the present work comes from optimal control. In fact, several of the recent results mentioned before were derived in an optimal control context; this is also true for [12]. Indeed, the analysis of a PDE-constrained optimization problem is usually based on detailed regularity and stability results for the underlying PDE, in particular also on global-in-time existence and uniqueness

in the first place. Obtaining the necessary results can be challenging especially in case of so-called *state constraints*, that is, pointwise upper and lower bounds on the solution u (the *state*) in every $(t, x) \in \bar{Q}$. The difficulty arises here because the amount of regularity for the state required to deal with this type of constraint is relatively high; for instance, rigorously establishing first-order necessary optimality conditions in this context via the standard technique requires *continuous* solutions of the state equation, see e.g. [11]. For quasilinear parabolic problems such as (1) this regularity is guaranteed within the setting of [9, 43], and one can proceed with the usual reasoning, see [32, 41]. In fact, in this case, continuity of the solution is a crucial aspect of the state equation analysis in the first place as explained above. Nevertheless, the derivation of second-order sufficient conditions for the same type of problem usually requires more regularity for conceptual reasons, at least for the linearized state equation; this is a reason why the authors of [13] consider a more regular geometric setting. Moreover, for even more demanding types of constraints, such as pointwise bounds on the *gradient* of the state, one requires even more regularity for the solution to (1) such as $\nabla u \in C(\bar{Q}, \mathbb{R}^d)$ (Such a constraint would be used e.g. to force material stresses to lie within certain bounds.) In this paper, we thus strive for the optimal regularity obtainable in dependence on the data within the scale X_θ of function spaces.

Outline and organization. As already mentioned above, this work contains several new contributions which we collect in the following milestones: First, we revisit and reprove the global-in-time existence result for $X_0 = W_D^{-1,p}(\Omega)$ in [43, Theorem 5.3] to also incorporate up to linear growth in the nonlinearity. This is Theorem 3.1. Second, we extend the nonautonomous maximal parabolic regularity result for $-\nabla \cdot \eta \mu \nabla + 1$ obtained in [9] for $X = H_D^{-\zeta,p}(\Omega)$ to the whole scale X_θ using Hölder-continuity of the scalar coefficient perturbation η of degree $> \theta$, see Theorem 4.3. Our proof here is also much less involved than the one in [9] since we establish an invariance property for the domain of the elliptic operators with respect to η . This result then allows to, third, bootstrap the global-in-time solution u from the ambient space X_0 to X_θ with the associated regularity $u \in W^{1,s}(0, T; \text{Dom}_{X_\theta}(-\nabla \cdot \mu \nabla + 1)) \cap L^s(0, T; X_\theta)$, up to a certain threshold $\bar{\theta}$. We have split this result into two parts, Theorems 5.7 and 5.10. If the solutions to elliptic problem with mixed boundary conditions associated to $-\nabla \cdot \mu \nabla + 1$ in $L^p(\Omega)$ are at least Lipschitz-continuous, then our procedure works up to $X_1 = L^p(\Omega)$ indeed. (A particular case would be that of optimal Sobolev regularity $W^{2,p}(\Omega) \cap W_D^{1,p}(\Omega)$ for the elliptic problem in $L^p(\Omega)$.) In this sense, our results close the gap between existence and uniqueness of global-in-time solutions for the $X_0 = W_D^{-1,p}(\Omega)$ - and the $X_1 = L^p(\Omega)$ setting as obtained in [43] and [12], respectively.

To reiterate, our overall strategy to obtain global-in-time solutions to (1) in X_θ can be summarized as follows: We obtain a global-in-time solution for $\theta = 0$, so in $X_0 = W_D^{-1,p}(\Omega)$. Re-inserting this solution into the nonlinear functions, we see that the solution satisfies a linear nonautonomous problem whose coefficient $\xi(u)$ admits a certain degree of Hölder-continuity. This degree of Hölder-continuity ϑ allows to invoke nonautonomous maximal parabolic regularity in X_θ for $\theta < \vartheta$, from which we obtain that the solution u is, in truth, more regular. But, repeating the argument, we see that improved regularity of u has also improved the degree of Hölder-continuity to $\vartheta^+ > \vartheta$, and we obtain even better regularity for u than before. This procedure works iteratively up to some maximal choice $\bar{\theta}$ of θ that ultimately depends on the domain, the boundary conditions and μ , via the maximal degree

1 of Hölder continuity for solutions to the elliptic problem with mixed boundary
 2 conditions associated to $-\nabla \cdot \mu \nabla + 1$ in $L^p(\Omega)$.

3 The paper is organized as follows: In Section 2, we introduce notation and as-
 4 sumptions and collect several auxiliary concepts and results used throughout the
 5 paper. We also formulate equation (1) in an appropriate function space setting, see
 6 Section 2.5. In Section 3.1, we provide the global-in-time existence and uniqueness
 7 result for $X_0 = W_D^{-1,p}(\Omega)$ in Theorem 3.1. Moreover, we collect the improved reg-
 8 ularity results on $H_D^{-\zeta,p}(\Omega)$ from [9] and use this to give a result for $X = L^{p/2}(\Omega)$,
 9 too. The main step towards the overall main result is then obtained in Section 4,
 10 where we establish nonautonomous maximal parabolic regularity for $-\nabla \cdot \eta \mu \nabla + 1$
 11 in X_θ when η is Hölder-continuous in space of degree $> \theta$; this is Theorem 4.3. The
 12 fundamental achievement here is that the domains of the foregoing operators in X_θ
 13 coincide with that of $-\nabla \cdot \mu \nabla + 1$, so, they are invariant under η . This insight is
 14 based on bilinear interpolation as in Appendix A.1. Finally, in Section 5 we tackle
 15 global-in-time existence for (1) in X_θ by employing the nonautonomous maximal
 16 regularity result from Section 4. More precisely, we first collect some results on
 17 interpolation and domains of fractional powers of $-\nabla \cdot \mu \nabla + 1$ in Section 5.1; these
 18 are required for the internal workings of the bootstrapping machinery. A first ver-
 19 sion of the main result, Theorem 5.7, is then provided in Section 5.2, to make the
 20 argument more transparent. The second and more comprehensive version is given
 21 afterwards in Section 5.3 via Theorem 5.10. A discussion of particular cases where
 22 we can determine the upper threshold $\bar{\theta}$ for θ more or less explicitly can be found
 23 in Appendix B.

24 **2. Preliminaries.** In this section we introduce several notions and definitions. We
 25 further collect several auxiliary results which will be used throughout the paper.
 26 Finally, once the necessary groundwork has been done, we state the standing as-
 27 sumptions for this work as well as the abstract quasilinear parabolic PDE problem
 28 in more detail.

29 **2.1. Notation and conventions.** We fix the given time interval $I = (0, T)$ with
 30 $T > 0$. All vector spaces considered are *real* ones. The domain of a closed operator
 31 $A: X \rightarrow Y$ between Banach spaces X, Y is denoted by $\text{Dom}_X(A)$; in general, we
 32 equip it with the graph norm. If A is bijective, then $x \mapsto \|Ax\|_Y$ is equivalent to
 33 the graph norm. If A is a closed operator in X and $Z \subseteq X$, then we denote the
 34 domain of the corestriction of A to Z by $\text{Dom}_Z(A) = \{x \in X: Ax \in Z\}$. Moreover,
 35 by $\mathcal{L}(X, Y)$ we refer to the space of bounded linear operators $X \rightarrow Y$ with the
 36 operator norm. By $\hookrightarrow_c$ and $\hookrightarrow_d$ we denote compact and dense embedding. We
 37 use standard notation and definitions for classical Lebesgue- and Hölder function
 38 spaces, also for Bochner-Lebesgue $L^s(J, X)$ and Bochner-Sobolev $W^{1,s}(J, X)$ spaces
 39 on an interval, and for real $(\cdot, \cdot)_{\theta,p}$ and complex $[\cdot, \cdot]_\theta$ interpolation spaces. Usually,
 40 we will omit the underlying spatial domain Ω in the associated function spaces,
 41 if no confusion is likely and no particular point is to be made. Finally, given an
 42 integrability exponent $p \in [1, \infty]$ we denote by $p' \in [1, \infty]$ the conjugate exponent
 43 defined by $1/p + 1/p' = 1$.

44 **2.2. Domain and function spaces.** We next introduce the assumptions on the
 45 geometry of the underlying domain Ω and its boundary parts Γ_N and Γ_D , followed
 46 by some more function spaces for which this geometry is (partially) important. Here
 47 is the standing assumption on the domain:

Assumption 2.1 (Geometry). Let $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$, be a bounded domain with boundary $\partial\Omega$; let $\Gamma_N \subseteq \partial\Omega$ be relatively open, denoting the designated Neumann boundary part, and let $\Gamma_D = \partial\Omega \setminus \Gamma_N$ denote the Dirichlet boundary part. We assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger, that is, a weak Lipschitz domain (“Lipschitz manifold”) with a compatibility condition for $\overline{\Gamma_N} \cap \Gamma_D$, see [23]. Further, we require the additional property that every Lipschitz chart in the definition of regular in the sense of Gröger can be chosen to be volume-preserving.

We will not need the precise technical formulation for the notions in Assumption 2.1 which is why we refer to [23]; see also [24] for a 3D characterization. However, a few comments are in order:

- (1) Note that we do *not* assume $\partial\Omega$ to (locally) be the graph of a Lipschitz function as it is required in the case of a *strong* Lipschitz domain, cf. [21, Definition 1.2.1.1]. Any such strong Lipschitz domain will be a weak Lipschitz domain with volume-preserving charts. In particular, it will immediately be regular in the sense of Gröger if either $\Gamma_N = \emptyset$ or $\Gamma_N = \partial\Omega$, see [25, Remark 3.3]. The classical example of a weak Lipschitz domain which is not also a strong Lipschitz domain is that of a pair of crossing beams in 3D [25, Section 7.3].
- (2) The assumption that the Lipschitz charts be volume-preserving is posed *only* in order to be able to utilize results from [43] where this property was used in order to avoid technical particularities in a localization/transformation procedure. Thus, one could get rid of this particular assumption by re-visiting and improving upon the corresponding results in [43].

Function spaces. We next turn to some more function spaces which require saying a bit more in their definition and for which the geometry of Ω is important. First, for $s \in \mathbb{R}$ and $q \in (1, \infty)$, let $H^{s,q}(\mathbb{R}^d)$ be the classical Bessel potential spaces. In order to incorporate a type of zero trace property into the function spaces, we note that for $s \in (1/q, 1 + 1/q)$, there exists a continuous linear *trace operator* $\text{tr}_D: H^{s,q}(\mathbb{R}^d) \rightarrow L^q(\Gamma_D)$, see [33, Theorems VI.1&VII.1]. Here and in all what follows, the Lebesgue space on a subset of the boundary $\partial\Omega$ is equipped with the $(d-1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}$. For such s , define the Bessel potential spaces on $\mathbb{R}^d$ incorporating a homogeneous Dirichlet condition on Γ_D by

$$H_D^{s,q}(\mathbb{R}^d) := \left\{ f \in H^{s,q}(\mathbb{R}^d) : \text{tr}_D f \equiv 0 \right\}.$$

Since the trace operator is continuous on $H^{s,q}(\mathbb{R}^d)$, its kernel $H_D^{s,q}(\mathbb{R}^d)$ is a closed subspace and thus a Banach space.

The corresponding function spaces on Ω for $s \geq 0$ and $s \in (1/q, 1 + 1/q)$, respectively, are now defined by restriction:

$$H^{s,q}(\Omega) := \left\{ f|_\Omega : f \in H^{s,q}(\mathbb{R}^d) \right\} \quad \text{and} \quad H_D^{s,q}(\Omega) := \left\{ f|_\Omega : f \in H_D^{s,q}(\mathbb{R}^d) \right\},$$

and equipped with the canonical quotient norms. This makes them Banach spaces. We moreover set $H^{-s,q}(\Omega) := (H^{s,q'}(\Omega))^*$ and $H_D^{-s,q}(\Omega) := (H_D^{s,q'}(\Omega))^*$. Then all the spaces introduced so far are reflexive. Note also that $H^{0,q}(\Omega) = L^q(\Omega)$. The quotient spaces so far are very abstract. It will however turn out that they can be related to the usual, intrinsically defined Sobolev spaces due to Assumption 2.1.

Indeed, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we define the classical Sobolev spaces $W^{k,q}(\Omega)$ in the canonical way, that is, the set of $L^q(\Omega)$ functions whose distributional derivatives up to order k are regular and represented by $L^q(\Omega)$ functions, with the ℓ^q -type

norm. Let further

$$C_D^\infty(\mathbb{R}^d) := \left\{ f \in C_c^\infty(\mathbb{R}^d) : \text{dist}(\text{supp } f, \Gamma_D) > 0 \right\}, \quad C_D^\infty(\Omega) := C_D^\infty(\mathbb{R}^d)|_\Omega$$

1 and set $W_D^{1,q}(\Omega) := \overline{C_D^\infty(\Omega)}^{W^{1,q}(\Omega)}$ and $W_D^{-1,q}(\Omega) := (W_D^{1,q'}(\Omega))^*$. Then, in our
 2 geometric setting as in Assumption 2.1—and, in fact, also quite far beyond that—,
 3 we have $W_D^{1,q}(\Omega) = H_D^{1,q}(\Omega)$ for $q \in (1, \infty)$, up to equivalent norms since there is a
 4 suitable extension operator for $W_D^{1,q}(\Omega)$ to $W^{1,q}(\mathbb{R}^d)$ ([6, Proposition B.3]).

5 By construction, we have the usual Sobolev embeddings within the $H^{s,q}(\Omega)$ fam-
 6 ily at hand since these transfer immediately from $H^{s,q}(\mathbb{R}^d)$ to the quotient type
 7 spaces. This includes Rellich-Kondrachov type compactness results. In particular,
 8 they also hold for $W_D^{1,q}(\Omega)$ by virtue of the last paragraph, and also for embeddings
 9 into the Hölder spaces which we state for further use.

10 **Lemma 2.2** (Sobolev). *Let $q \in (1, \infty)$ and $s > \frac{d}{q}$. Then $H^{s,q}(\Omega) \hookrightarrow C^{0,s-\frac{d}{q}}(\overline{\Omega})$.*

11 The following interpolation result based on main results in [6] and the duality
 12 principle for complex interpolation [8, Corollary 4.5.2] will be very useful. It shows
 13 that the Bessel scale on Ω is indeed an interpolation scale. We point out that the
 14 present Assumption 2.1 on the geometry of Ω implies the assumptions in [6], see
 15 the introduction there, cf. also [9, Appendix A].

Lemma 2.3 ([6, Theorems 1.1&1.3]). *Let $q \in (1, \infty)$ and $\theta \in (0, 1)$. Then we have*

$$\left[L^q(\Omega), W_D^{\pm 1,q}(\Omega) \right]_\theta = \begin{cases} H_D^{\pm \theta, q}(\Omega) & \text{if } \theta > \frac{1}{q}, \\ H^{\pm \theta, q}(\Omega) & \text{if } \theta < \frac{1}{q} \end{cases}$$

up to equivalent norms. Moreover,

$$\left[W_D^{-1,q}(\Omega), W_D^{1,q}(\Omega) \right]_{\frac{1}{2}} = L^q(\Omega)$$

16 again up to equivalent norms.

Corollary 2.4. *Let $q > d$ and $\theta \in (\frac{1}{2} + \frac{d}{2q}, 1]$. Then*

$$\left[W_D^{-1,q}(\Omega), W_D^{1,q}(\Omega) \right]_\theta \hookrightarrow C^{0,2\theta-1-\frac{d}{q}}(\overline{\Omega}).$$

Proof. The condition on θ requires exactly that $2\theta - 1 > \frac{d}{q}$. Thus, with the reiter-
 ation theorem for complex interpolation ([54, Remark 1.9.3.1]) and Lemma 2.3:

$$\begin{aligned} \left[W_D^{-1,q}(\Omega), W_D^{1,q}(\Omega) \right]_\theta &= \left[\left[W_D^{-1,q}(\Omega), W_D^{1,q}(\Omega) \right]_{\frac{1}{2}}, W_D^{1,q}(\Omega) \right]_{2\theta-1} \\ &= \left[L^q(\Omega), W_D^{1,q}(\Omega) \right]_{2\theta-1} = H_D^{2\theta-1,q}(\Omega) \end{aligned}$$

17 and the claim follows by Sobolev embedding (Lemma 2.2). $\square$

18 Finally, let us note that by Assumption 2.1, Ω is a (weak) Lipschitz domain.
 19 Thus, there exists a well defined trace operator tr which is continuous

$$\text{tr}: H_D^{\theta,q}(\Omega) \rightarrow L^q(\partial\Omega) \quad \text{for } \theta \in (\tfrac{1}{q}, 1] \quad (2)$$

20 and for which $\text{tr } u = u|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega}) \cap H_D^{\theta,q}(\Omega)$. This follows for example
 21 using a multiplicative trace inequality as in [39, Corollary 1.4.7.1] and Lemma 2.3,
 22 see [25, Lemma 3.6]. The condition $\theta > \frac{1}{q}$ is sharp to have a trace operator. Further,
 23 we denote by tr_N the natural restriction to Γ_N , that is, $\text{tr}_N u := (\text{tr } u)|_{\Gamma_N}$. Then,
 24 by duality, we immediately have:

1 **Lemma 2.5.** *Let $q \in (1, \infty)$ and let $\theta \in [0, \frac{1}{q})$. Then*

$$\mathrm{tr}_N^*: L^q(\Gamma_N) \rightarrow [W_D^{-1,q}(\Omega), L^q(\Omega)]_\theta = H_D^{\theta-1,q}(\Omega)$$

2 *is continuous.*

3 **2.3. Maximal parabolic regularity.** We next turn to the notion of maximal
 4 parabolic regularity. The case considered here will be that of *constant domains*.
 5 Given two Banach spaces $E_1 \hookrightarrow_d E_0$, we will use the following abbreviation for the
 6 maximal regularity type spaces:

$$\mathbb{W}^{1,r}(I, (E_0, E_1)) := W^{1,r}(I, E_0) \cap L^r(I, E_1), \quad r \in (1, \infty).$$

7 Suppose that there exists an operator on E_0 with domain E_1 which is the generator
 8 of an analytic semigroup on E_0 . (This will always be satisfied in the following.)
 9 Let $A: I \rightarrow \mathcal{L}(E_1, E_0)$ be a bounded and measurable operator family such that
 10 $A(t)$ is a closed operator in E_0 with domain E_1 for each $t \in I$. Then A is said
 11 to satisfy *(nonautonomous) maximal parabolic regularity* on $L^r(I, E_0)$, if for every
 12 $f \in L^r(I, E_0)$ and every $w_0 \in (E_0, E_1)_{1/r', r}$ there exists a unique solution $w \in$
 13 $\mathbb{W}^{1,r}(I, (E_0, E_1))$ to the equation

$$\partial w + Aw = f \quad \text{in } L^r(I, E_0), \quad w(0) = w_0 \quad \text{in } (E_0, E_1)_{1/r', r}$$

14 where $\partial: W^{1,r}(I, E_0) \rightarrow L^r(I, E_0)$ denotes the distributional derivative. (We tac-
 15 itly identify $(Aw)(t) = A(t)w(t)$ here.) Equivalently, A satisfies (nonautonomous)
 16 maximal parabolic regularity on $L^r(I, E_0)$ *if and only if* the total differential oper-
 17 ator

$$(\partial + A, \gamma_0): \mathbb{W}^{1,r}(I, (E_0, E_1)) \rightarrow L^r(I, E_0) \times (E_0, E_1)_{1/r', r} \quad (3)$$

18 is continuously invertible. Due to the assumption on E_0 and E_1 , it is also equivalent
 19 to consider only the case of initial value 0. We refer to e.g. [1, Proposition 3.1].

20 If A is in fact autonomous, that is, $A(t) \equiv A$ for every $t \in I$, then maximal
 21 parabolic regularity of A on $L^\rho(I, E_0)$ for *some* $\rho \in (1, \infty)$ is equivalent to maximal
 22 parabolic regularity of A on $L^r(I, E_0)$ for *any* $r \in (1, \infty)$ ([1, Remark 6.1d]). In
 23 this case we just say that A satisfies *maximal parabolic regularity on E_0* .

24 We will freely use that if A satisfies maximal parabolic regularity on E_0 , then so
 25 does $A + \lambda$ for any scalar λ .

26 The following proposition with a sufficient condition for nonautonomous maximal
 27 parabolic regularity going back to [1, 48] will be the driving force for our later
 28 considerations:

29 **Proposition 2.6** ([1, 48]). *In the above setting, suppose that $A: \bar{I} \rightarrow \mathcal{L}(E_1, E_0)$ is*
 30 *continuous and that for every $\tau \in \bar{I}$, the operator $A(\tau)$ satisfies maximal parabolic*
 31 *regularity on E_0 . Then A satisfies nonautonomous maximal parabolic regularity on*
 32 *$L^r(I, E_0)$ for every $r \in (1, \infty)$.*

33 Since an operator satisfying maximal parabolic regularity on E_0 is also the (neg-
 34 ative) generator of an analytic semigroup on E_0 with domain E_1 , the above as-
 35 sumption in this regard is always satisfied in the context of Proposition 2.6.

36 We close this section with embeddings for the maximal regularity spaces.

Lemma 2.7 ([2, Theorem 3]). *Let $E_1 \hookrightarrow_d E_0$ be as above and let $r \in (1, \infty)$. Then*

$$\mathbb{W}^{1,r}(I, (E_0, E_1)) \hookrightarrow C(\bar{I}, (E_0, E_1)_{1/r', r}).$$

Further, if $\theta \in [0, 1 - \frac{1}{r})$ and $\tau \in (\frac{1}{r}, 1 - \theta)$, then

$$\mathbb{W}^{1,r}(I, (E_0, E_1)) \hookrightarrow C^{0, \tau-1/r}(\bar{I}, [E_0, E_1]_\theta),$$

1 and the latter embedding is in fact compact when $E_1 \hookrightarrow_c E_0$.

2.4. **The differential operator.** We say that $\rho: \Omega \rightarrow \mathbb{R}^{d \times d}$ is a *coefficient function* if it is measurable, bounded, and coercive in the sense that

$$\frac{z \cdot \rho(x) z}{|z|^2} \geq \rho_\bullet > 0 \quad \text{for almost all } x \in \Omega \quad \text{for all } z \in \mathbb{R}^d.$$

If ρ is a coefficient function, then we define the second-order divergence form operator

$$-\nabla \cdot \rho \nabla: W_D^{1,2}(\Omega) \rightarrow W_D^{-1,2}(\Omega), \quad \langle -\nabla \cdot \rho \nabla u, v \rangle := \int_{\Omega} \rho \nabla u \cdot \nabla v \quad (v \in W_D^{1,2}(\Omega)).$$

2 By the assumptions on the coefficient function ρ , the operator $-\nabla \cdot \rho \nabla$ is continuous.
 3 Further, due to the Lax-Milgram Lemma, $-\nabla \cdot \rho \nabla + 1$ is a (topological) isomorphism
 4 and we can omit the “+1” if $\Gamma_D \neq \emptyset$. In the present geometric framework of
 5 Assumption 2.1, we then automatically have $\mathcal{H}_{d-1}(\Gamma_D) > 0$. Consider the part of
 6 the operator in $W_D^{-1,q}(\Omega)$ for $q > 2$. We do not relabel this operator by slight abuse
 7 of notation; it will always be clear from the context which q is meant. This operator
 8 is clearly still bijective, but in general we will *not* have $\text{Dom}_{W_D^{-1,q}(\Omega)}(-\nabla \cdot \rho \nabla + 1) =$
 9 $W_D^{1,q}(\Omega)$; at least not for q which are not very close to 2 ([26, Theorem 5.6]).

If $\text{Dom}_{W_D^{-1,q}(\Omega)}(-\nabla \cdot \rho \nabla + 1) = W_D^{1,q}(\Omega)$, a situation which we will enforce for a
 $q > d$ as one of our main assumptions below, then there are some good consequences.
 To set the stage, we first introduce, for any $q \geq 2$, the part A of $-\nabla \cdot \rho \nabla$ in $L^q(\Omega)$
 by

$$\begin{aligned} \text{Dom}(A) &= \left\{ u \in W_D^{1,2}(\Omega) \cap L^q(\Omega) : -\nabla \cdot \rho \nabla u \in L^q(\Omega) \right\}, \\ Au &= -\nabla \cdot \rho \nabla u \quad (u \in \text{Dom}(A)). \end{aligned}$$

10 Again, it will always be clear from context to which q the current incarnation of
 11 A refers. In the general context of mixed boundary conditions and an irregular
 12 domain, the domain of A will be very difficult to determine.

13 We next collect a few important properties of the operators A and $-\nabla \cdot \rho \nabla$. In
 14 the next result, the notion of *positive* operator is meant as in [54, Definition 1.14.1],
 15 but it is not fundamental for the rest of this paper.

16 **Proposition 2.8** ([5, Proposition 4.6/Theorem 11.5]). *Let ρ be a coefficient func-*
 17 *tion and let $q \geq 2$. Then we have the following:*

- 18 (1) *The operators A and $-\nabla \cdot \rho \nabla$ are positive operators on $L^q(\Omega)$ and $W_D^{-1,q}(\Omega)$.
 19 In particular, their fractional powers are well-defined.*
- 20 (2) *Even more, the operators admit a bounded $\mathcal{H}^\infty$ calculus. This implies that they
 21 exhibit bounded imaginary powers and maximal parabolic regularity.*

22 *The assertions for A also hold true for $q \in (1, 2)$.*

23 We can transfer the maximal parabolic regularity property also to the interpola-
 24 tion spaces between $W_D^{-1,q}(\Omega)$ and $L^q(\Omega)$. (In fact, the same is true for the bounded
 25 imaginary powers; we use this in Appendix A.)

26 **Corollary 2.9** ([25, Theorem 5.16iv]). *Let ρ be a coefficient function and let $q \geq 2$.
 27 Then the part of $-\nabla \cdot \rho \nabla$ in $[W_D^{-1,q}(\Omega), L^q(\Omega)]_\theta$ satisfies maximal parabolic regularity
 28 for every $\theta \in [0, 1]$.*

1 Proposition 2.8 enables us in particular to talk about the *square root* of the
2 associated operators. For $q > 2$, if in fact

$$\text{Dom}_{W_D^{-1,q}(\Omega)}(-\nabla \cdot \rho \nabla + 1) = W_D^{1,q}(\Omega), \quad (4)$$

3 or, equivalently, $-\nabla \cdot \rho \nabla + 1$ is a topological isomorphism $W_D^{1,q}(\Omega) \rightarrow W_D^{-1,q}(\Omega)$,
4 then for the square roots we have the following fundamental property at hand, the
5 *Kato square root property*:

Proposition 2.10 ([18, Theorem 6.5]). *Let ρ be a coefficient function and let $q \geq 2$. Suppose that (4) holds true. Then $\text{Dom}(A + 1)^{1/2} = W_D^{1,q}(\Omega)$, that is,*

$$(A + 1)^{1/2} : W_D^{1,q}(\Omega) \rightarrow L^q(\Omega) \quad \text{is a topological isomorphism.}$$

6 The *Kato square root property* classically refers to the case $q = 2$ which is the
7 fundamental basis for the remaining ones; we refer to the seminal work [7], see
8 also [5]. It is always satisfied if μ is *symmetric*. Note that from Proposition 2.10 it
9 also follows that $(-\nabla \cdot \rho \nabla + 1)^{1/2}$ is a topological isomorphism $L^q(\Omega) \rightarrow W_D^{-1,q}(\Omega)$,
10 cf. [5, Theorem 11.5].

From the following sketch argument we infer that we can equivalently regard
 $(A + 1)^{1/2}$ either as the square root of $A + 1$ or as the part of the square root
 $(-\nabla \cdot \rho \nabla + 1)^{1/2}$ in $L^q(\Omega)$:

$$\begin{aligned} (-\nabla \cdot \rho \nabla + 1)^{-1/2} L^q(\Omega) &= (-\nabla \cdot \rho \nabla + 1)^{-1} (-\nabla \cdot \rho \nabla + 1)^{1/2} L^q(\Omega) \\ &= (-\nabla \cdot \rho \nabla + 1)^{-1} W_D^{-1,q}(\Omega) \\ &= W_D^{1,q}(\Omega) = (A + 1)^{-1/2} L^q(\Omega). \end{aligned}$$

With similar reasoning based on the foregoing, we in fact obtain the same property
for all fractional powers:

$$\text{Dom}_{L^q(\Omega)}(-\nabla \cdot \rho \nabla + 1)^\gamma = \text{Dom}(A + 1)^\gamma \quad (\gamma \in [0, 1]).$$

11 This will essentially allow us to get rid of the operator A in the following consid-
12 erations, which eases notation significantly. Finally, we will make free use of the
13 *reiteration theorem* for fractional powers ([54, Theorem 1.15.3]), owing to A being
14 positive and admitting bounded imaginary powers as established in Proposition 2.8:

Lemma 2.11. *Let ρ be a coefficient function and let $q \geq 2$. Suppose that (4) holds true. If $\gamma = (1 - \theta)\alpha + \theta\beta$ for $\alpha, \beta \in [0, 1]$ and $\theta \in (0, 1)$, then*

$$\text{Dom}_{L^q}(-\nabla \cdot \rho \nabla + 1)^\gamma = \left[\text{Dom}_{L^q}(-\nabla \cdot \rho \nabla + 1)^\alpha, \text{Dom}_{L^q}(-\nabla \cdot \rho \nabla + 1)^\beta \right]_\theta.$$

15 We gather a permanence principle for the optimal elliptic regularity property (4).
16 The upper bound $q \leq 6$ in the statement is a technical limitation related to the
17 Sobolev exponent $2^* = 6$ in $d = 3$.

Lemma 2.12 ([19, Lemma 6.2]). *Let ρ be a coefficient function and let $q \in [2, 6]$. Suppose that (4) holds true. Then*

$$\text{Dom}_{W_D^{-1,q}(\Omega)}(-\nabla \cdot \eta \rho \nabla + 1) = W_D^{1,q}(\Omega)$$

18 for every uniformly continuous positive function $\eta \in C(\bar{\Omega})$.

Finally, for $\varrho \in L^\infty(\Gamma_N)$, we will use $\mathcal{B}_\varrho := \text{tr}_N^* \circ [\varrho \text{tr}_N]$ to signify the mapping associated to a Robin boundary condition. Indeed, for $\theta \in (1 - \frac{1}{q}, 1]$, the following operator is well-defined and continuous due to (2) and Lemma 2.5:

$$\langle \mathcal{B}_\varrho u, v \rangle := \int_{\Gamma_N} \varrho (\text{tr}_N u) (\text{tr}_N v), \quad \mathcal{B}_\varrho: W_D^{1,q}(\Omega) + C(\overline{\Omega}) \rightarrow H_D^{-\theta,q}(\Omega).$$

2.5. **Problem statement and assumptions.** Next, we state the minimal assumptions on the data of (1) that will allow us to obtain global-in-time solutions of (1) in the $X_0 = W_D^{-1,p}$ -setting. These global-in-time solutions will be the starting point for the bootstrap procedure through the scale $X_\theta = [L^q(\Omega), W_D^{-1,q}(\Omega)]_\theta$ in Section 4. Of course, it will be necessary to assume more for some objects—more precisely, for $s, \mathcal{F}$ and u_0 —to improve regularity later on.

Assumption 2.13. Suppose that the following properties hold true:

- (Co) The function $\xi: \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous and satisfies $0 < \xi_\bullet \leq \xi(z) \leq \xi^\bullet$ for all $z \in \mathbb{R}$. Further, μ is a coefficient function. For the coefficient in the Robin boundary condition we suppose $\alpha \in L^\infty(I, L^\infty(\Gamma_N))$.
- (Iso) We assume that there is $p \in (d, d+1]$ such that

$$-\nabla \cdot \mu \nabla + 1: W_D^{1,p}(\Omega) \rightarrow W_D^{-1,p}(\Omega)$$

is a topological isomorphism and **fix this choice** of p . Let further $\frac{1}{s} < \frac{1}{2} - \frac{d}{2p}$.

- ($\mathcal{F}$) $\mathcal{F} := \mathcal{F}_\Omega + \text{tr}_N^* \mathcal{F}_\Gamma: I \times (W_D^{-1,p}, W_D^{1,p})_{1/s',s} \rightarrow W_D^{-1,p}$ is a locally Lipschitz Caratheodory map, that is, it is measurable with respect to the first variable, continuous with respect to the second, and for every $R > 0$ there is $L_R \in L^s(I)$ such that for almost all $t \in I$:

$$\|\mathcal{F}(t, w_1) - \mathcal{F}(t, w_2)\|_{W_D^{-1,p}} \leq L_R(t) \|w_1 - w_2\|_{(W_D^{-1,p}, W_D^{1,p})_{1/s',s}},$$

whenever $w_1, w_2 \in (W_D^{-1,p}, W_D^{1,p})_{1/s',s}$ and $\|w_i\|_{(W_D^{-1,p}, W_D^{1,p})_{1/s',s}} \leq R$, $i = 1, 2$.

Moreover, $\mathcal{F}$ obeys an up-to-linear growth condition in the second variable: There is $\psi \in L^s(I)$ such that for all $w \in (W_D^{-1,p}, W_D^{1,p})_{1/s',s}$ and almost all $t \in I$,

$$\|\mathcal{F}(t, w)\|_{W_D^{-1,p}} \leq \psi(t)(1 + \|w\|_{C(\overline{\Omega})}).$$

- (IV) The initial condition satisfies $u_0 \in (W_D^{-1,p}, W_D^{1,p})_{1/s',s}$.

The setting described in Assumptions 2.1 and 2.13 is similar to the one considered in [43, Section 5]. The major difference is that we now allow the function $\mathcal{F}$ to grow up to linearly with respect to the function variable instead of imposing a quite harsh global boundedness assumption as done in [43].

Still, Assumptions 2.1 and 2.13, in particular (Iso), impose non-trivial conditions on the considered setting. A few comments are thus in order.

Remark 2.14. (1) If Ω is a bounded domain with Lipschitz boundary (strong Lipschitz domain) and $\Gamma_N = \emptyset$ or $\Gamma_N = \partial\Omega$ and μ is a symmetric and uniformly continuous coefficient function, then there will be $p > 3$ such that (Iso) in Assumption 2.13 is satisfied, see [20, Theorem 3.12, Remark 3.17]. In this sense, Assumptions 2.1 and 2.13 cover the classical “regular” setting of strong Lipschitz domains in spatial dimensions $d = 2, 3$ with pure Dirichlet or Neumann boundary conditions and a symmetric, uniformly continuous coefficient

- 1 function. Further, from the pioneering work of Gröger [23] we find that As-
 2 sumption 2.13 (Iso) is *always* satisfied for some $p > 2$ within our setting. In
 3 fact, this is also true under much more general assumptions on the geometry of
 4 the domain, see [26].
- 5 (2) It is well known that in the presence of mixed boundary conditions, the iso-
 6 morphism property in Assumption 2.13 (Iso) can only be expected to hold for
 7 some $p < 4$ in general due to the Shamir counterexample [51, Introduction].
 8 Moreover, if Assumption 2.13 (Iso) is valid, then it is also valid for all $q \in [2, p]$
 9 due to interpolation and the Lax-Milgram Lemma. In this sense, the upper
 10 bound of $d + 1$ in (Iso) should not be considered as critical. We pose it for tech-
 11 nical reasons to avoid some case distinctions. Nevertheless, the authors of [19]
 12 establish a rich zoo of real-world constellations such that Assumption 2.13 (Iso)
 13 is satisfied in dimension $d = 3$ within the constraints of the other assumptions.

Remark 2.15. We point out that the growth condition in Assumption 2.13 ($\mathcal{F}$) is indeed well-defined since

$$(W_D^{-1,p}, W_D^{1,p})_{1/s',s} \hookrightarrow C(\overline{\Omega}).$$

14 In fact, choose $\frac{1}{2} + \frac{d}{2p} < \theta < 1 - \frac{1}{s}$. Then $(W_D^{-1,p}, W_D^{1,p})_{1/s',s} \hookrightarrow [W_D^{-1,p}, W_D^{1,p}]_\theta$
 15 and Corollary 2.4 strikes. Such a growth condition is the standard requirement in
 16 the analysis of abstract semilinear equations, see for example [28, Corollary 3.3.5].

17 Besides rather obvious choices for $u \mapsto \mathcal{F}(u)$ such as a “constant” function $f \in$
 18 $L^s(I, W_D^{-1,p})$ and nonlinear Nemytskii operators induced by suitable real functions,
 19 we point to two particular possible incarnations of $\mathcal{F}$:

- 20 • Fix $g \in L^s(I, W_D^{1,p})$ —possibly coming from some other differential equation—
 21 and set $\mathcal{F}(t, w) := \nabla \cdot w \mu \nabla g(t)$. Such a drift-type term arises e.g. in the mod-
 22 elling of semiconductors and has been considered within a semilinear parabolic
 23 PDE in [42].
- 24 • We can also consider nonlocal-in-space interactions in $\mathcal{F}$ such as for example
 25 $\mathcal{F}(t, w) := \sigma \left(\int_\Omega k(t) w \, dx \right)$, where $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ is bounded and Lipschitz and k
 26 is a suitable kernel with $k(t) \in L^1(\Omega)$.

27 **Remark 2.16.** The assumption that ξ is uniformly bounded from below is neces-
 28 sary to ensure uniform ellipticity of the quasilinear differential operator and cannot
 29 be avoided easily. The upper bound on ξ , however, is not strictly necessary and
 30 can be removed utilizing a Stampacchia argument, cf. [12, Theorem 2.1].
 31 Such an argument does not rely on the stronger regularity assumptions posed in the
 32 cited work. Similarly, it is also possible to discuss nonlinear functions $\mathcal{F}$ that are
 33 monotone with respect to the second variable but not necessarily of linear growth
 34 such as the classical $\mathcal{F}(t, w) = w^3$. However, in order to keep the discussion more
 35 transparent we decided not to include these technical modifications.

Abstract problem formulation. We next give an abstract but precise formulation
 of (1). The goal is to find a *global-in-time* solution $u \in \mathbb{W}^{1,s}(I, (W_D^{-1,p}, W_D^{1,p}))$ to

$$\left. \begin{aligned} \partial u - \nabla \cdot \xi(u) \mu \nabla u + u + \mathcal{B}_\alpha u &= \mathcal{F}(u) && \text{in } L^s(I, W_D^{-1,p}), \\ u(0) &= u_0 && \text{in } (W_D^{-1,p}, W_D^{1,p})_{1/s',s} \end{aligned} \right\} \quad (5)$$

36 and to give sharp sufficient conditions along the scale $[W_D^{-1,p}, L^p]_\theta$ for when the
 37 solution is in fact more regular. We start in the $W_D^{-1,p}$ -setting ($\theta = 0$) because we

can in fact prove existence and uniqueness of a global-in-time solution there, basing on uniform Hölder estimates for nonautonomous parabolic evolution equations established in [43].

Herein, the Neumann/Robin boundary conditions of (1) have been absorbed into the distributional right hand side of (5) and $\mathcal{B}_\alpha u$, respectively, cf. also Assumption 2.13 ($\mathcal{F}$). The Dirichlet boundary conditions are prescribed by the underlying function space $W_D^{1,p}$. The above formulation is self-consistent in $L^s(I, W_D^{-1,p})$ for $u \in \mathbb{W}^{1,s}(I, (W_D^{-1,p}, W_D^{1,p}))$ due to the assumptions on the data in Assumption 2.13, cf. also Lemma 2.7 and Corollary 2.4.

As we ultimately plan to establish better regularity for u than claimed above, we introduce the following convention: Let subspaces $X \hookrightarrow W_D^{-1,p}$ and $Y \hookrightarrow W_D^{1,p}$ with $Y \hookrightarrow_d X$ and some $r \in (1, \infty)$ be given. Suppose that the solution u to (5) in fact satisfies $u \in \mathbb{W}^{1,r}(I, (X, Y))$, and that (5) holds true in $L^r(I, X) \times (X, Y)_{1/r', r}$. Then we say that u solves (5) (also) on X . Note that this solution will then necessarily be unique, since the one for $X = W_D^{-1,p}$ will be.

Remark 2.17. (1) In view of Lemma 2.5 on the adjoint trace operator and the scale $X_\theta = [W_D^{-1,p}, L^p]_\theta$, there is a natural threshold at $\theta = 1/p$ above which X_θ cannot accommodate any more distributional objects such as $\text{tr}_N^* g$ arising from inhomogeneous Neumann/Robin boundary data g . This also applies to $\mathcal{F}_\Gamma$.

(2) In fact, we will also allow for a Robin boundary condition in the abstract equation (5) in $[W_D^{-1,p}, L^p]_\theta$ only for $\theta < 1/p$. There, the boundary condition is enforced by adding the appropriate term $\mathcal{B}_\alpha$ and we can view $\mathcal{B}_\alpha$ as a perturbation of the main part of the differential operator $-\nabla \cdot \mu \nabla + 1$. This clearly does not work any more for $\theta > 1/p$ since in this (stronger) setting, the boundary condition is built into the differential operator in a strong sense. (This is a feature, not a bug.) However, it would be most desirable to also be able to incorporate a—then: homogeneous—Robin condition there. The problem is that we do lack an analogous result regarding the Kato square root property as in Proposition 2.10 for this case with $-\nabla \cdot \mu \nabla + 1 + \mathcal{B}_\alpha$. Since our later considerations, in particular in Section 5.1, are strongly based on Proposition 2.10, we are unable to accommodate $\alpha \neq 0$ at the moment. Any improvement in the square root property for the operator including $\mathcal{B}_\alpha$ would transfer to the present setting immediately.

3. Global-in-time solutions and regularity. In this section we provide existence and uniqueness of global-in-time solutions in the $W_D^{-1,p}$ - $W^{1,p}$ -setting for (5). This is the fundamental result on which our further considerations are based on since it delivers the global solution whose regularity we then can bootstrap. We further augment this result by briefly reviewing existing results on improved regularity in $X = H_D^{-\zeta, p}(\Omega)$ for ζ close to 1 and in $X = L^{p/2}(\Omega)$,

3.1. Existence of solutions. It follows our first main result that extends [43, Theorem 5.3]. The latter was proven using a “global” Schauder fixed-point argument based on the other main result in the paper, uniform Hölder estimates [43, Theorem 2.13], thereby requiring a rather strong global boundedness property for $\mathcal{F}$. Here, we rely on the same uniform Hölder estimates, but rather use them to disprove finite-time blowup of a local-in-time solution for which we can tolerate up to linear growth in $\mathcal{F}$.

1 In all what follows, we take Assumptions 2.1 and 2.13 for granted.

2 **Theorem 3.1.** *There is a unique global-in-time solution $u \in \mathbb{W}^{1,s}(I, (W_D^{-1,p}, W_D^{1,p}))$*
 3 *to (5).*

4 Before we start with the proof, let us note that there is $\beta > 0$ such that

$$\mathbb{W}^{1,s}(I, (W_D^{-1,p}, W_D^{1,p})) \hookrightarrow_c C^{0,\beta}(\bar{I}, C^{0,\beta}(\bar{\Omega})), \quad (6)$$

5 see Lemma 2.7 and Corollary 2.4.

6 *Proof.* We argue in a quite concise way how to obtain a local-in-time solution. Sim-
 7 ilar reasoning and associated arguments can be found in [43, Section 5] or [25, 40].
 8 From Corollary 2.4, we have $(W_D^{-1,p}, W_D^{1,p})_{1/s',s} \hookrightarrow C(\bar{\Omega})$. This has several con-
 9 sequences: For every w in the interpolation space, the operator $-\nabla \cdot \xi(w)\mu\nabla + 1$
 10 satisfies maximal parabolic regularity in $W_D^{-1,p}$ with domain $W_D^{1,p}$ due to Proposi-
 11 tion 2.8 and Lemma 2.12. Further, $w \mapsto -\nabla \cdot \xi(w)\mu\nabla + 1$ is Lipschitz continuous on
 12 bounded sets in $(W_D^{-1,p}, W_D^{1,p})_{1/s',s}$ with values in $\mathcal{L}(W_D^{1,p}, W_D^{-1,p})$, and $\mathcal{B}_{\alpha(t)}$ gives
 13 rise to a continuous linear operator $(W_D^{-1,p}, W_D^{1,p})_{1/s',s} \rightarrow W_D^{-1,p}$ for almost every
 14 $t \in I$, recall Lemma 2.5.

15 In (5), we transfer the Robin operator $\mathcal{B}_\alpha$ to the right-hand side $\bar{\mathcal{F}}(u) := \mathcal{F}(u) -$
 16 $\mathcal{B}_\alpha u$. From the foregoing considerations we infer that the assumptions of the
 17 seminal theorem of Prüss [47, Theorem 3.1] are satisfied for the resulting equa-
 18 tion, such that (5) admits a unique maximal local-in-time solution u in the max-
 19 imal regularity class. More precisely, there exists $T^\bullet \in (0, T)$ such that equa-
 20 tion (5) admits a unique solution u on $(0, T^\bullet)$ and for every $T_\bullet \in (0, T^\bullet)$ we have
 21 $u \in \mathbb{W}^{1,s}(0, T_\bullet, (W_D^{-1,p}, W_D^{1,p}))$. Thereby, $T^\bullet$ is characterized by the property that

$$\lim_{t \nearrow T^\bullet} u(t) \text{ does not exist in } (W_D^{-1,p}, W_D^{1,p})_{1/s',s}. \quad (7)$$

22 We show that in fact $T^\bullet = T$ with $u \in \mathbb{W}^{1,s}(0, T, (W_D^{-1,p}, W_D^{1,p}))$, using the linear
 23 growth assumption on $\mathcal{F}$ posed in Assumption 2.13 ($\mathcal{F}$) to disprove (7)

To this end, let $0 \leq \tau < T^\bullet$ be arbitrary for now—to be fixed later—and let
 $T_\bullet \in (\tau, T^\bullet)$. Let further $\mathcal{C}(\rho_\bullet, \rho^\bullet)$ be the set of all measurable nonautonomous
 coefficient functions $\rho: I \times \Omega \rightarrow \mathbb{R}^{d \times d}$ which are bounded in $L^\infty(I \times \Omega; \mathbb{R}^{d \times d})$ by
 $\rho^\bullet$ and which are uniformly coercive almost everywhere on $I \times \Omega$ with coercivity
 constant $\rho_\bullet$. Then, by the main result in [43, Theorem 2.13], for some $\gamma > 0$, w.l.o.g.
 $\gamma < \beta$ with β as in (6), the number

$$C_{\partial+A}(\tau, T_\bullet) := \sup_{\rho \in \mathcal{C}(\xi_\bullet, \mu_\bullet, \xi^\bullet, \mu^\bullet)} \|(\partial - \nabla \cdot \rho \nabla + 1)^{-1}\|_{\mathcal{L}(L^s(\tau, T_\bullet; W_D^{-1,p}); C_0^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega}))},$$

is well-defined and finite. Hereby, we denote by $(\partial - \nabla \cdot \rho \nabla + 1)^{-1}$ the solution
 operator $f \mapsto y$ of the nonautonomous problem

$$\left. \begin{aligned} \partial y - \nabla \cdot \rho \nabla y + y &= f && \text{in } L^s(I, W_D^{-1,p}), \\ y(\tau) &= 0 && \text{in } (W_D^{-1,p}, W_D^{1,p})_{1/s',s}, \end{aligned} \right\} \quad (8)$$

and the index 0 refers to zero initial condition at time τ for functions in $C_0^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})$. It is easy to see that we must have

$$C_{\partial+A}(\tau, T_\bullet) \leq C_{\partial+A}(0, T) =: C_{\partial+A}.$$

24 Indeed, every problem instance of (8) on $(\tau, T_\bullet)$ can be embedded into a problem
 25 instance of (8) on $(0, T)$ with the same input data size by extending and shifting
 26 the objects on $(\tau, T_\bullet)$ by zero. Then the definition of the operator norm gives the

- 1 estimate. This equicontinuity of the parabolic solution operator in the coercivity-
 2 and boundedness constants of the coefficient function will be the crucial element of
 3 the proof. Essentially, it will allow us to reason as in the semilinear case.

We need another global estimate, this time for the Robin operator whose mapping properties for almost every $t \in I$ were already mentioned above. Since α is assumed to be essentially bounded in time, we find (recall Lemma 2.5)

$$\begin{aligned} \|\mathcal{B}_\alpha\|_{\mathcal{L}(C^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega}), L^s(\tau, T_\bullet; W_D^{-1,p}))} \\ \leq |T_\bullet - \tau|^{\frac{1}{s}} \|\text{tr}\|_{\mathcal{L}(W_D^{1,p'}, L^{p'}(\Gamma_N))} \|\alpha\|_{L^\infty(I, L^\infty(\Gamma_N))} =: C_\alpha(\tau, T_\bullet). \end{aligned}$$

Now, in order to use the foregoing estimate with $C_{\partial+A}$, we split off the “initial value” $u(\tau)$ of the maximal solution u starting from τ via $w(t) := e^{-(t-\tau)\Delta} u(\tau)$. We have $w \in \mathbb{W}^{1,s}(\tau, T, (W_D^{-1,p}, W_D^{1,p}))$ by [3, Proposition III.4.10.2]. Set $v := u - w$ to obtain $v(\tau) = 0$. Then $v \in \mathbb{W}^{1,s}(\tau, T_\bullet, (W_D^{-1,p}, W_D^{1,p}))$ satisfies

$$\partial v - \nabla \cdot \xi(u) \mu \nabla v + v = \mathcal{F}(u) - \mathcal{B}_\alpha u - \partial w + \nabla \cdot \xi(u) \mu \nabla w - w, \quad v(\tau) = 0$$

in $L^s(\tau, T_\bullet; W_D^{-1,p})$. So, using the definition of $C_{\partial+A}$, the foregoing estimates, the embedding (6), and the linear growth assumption for $\mathcal{F}$ as in Assumption 2.13 ($\mathcal{F}$), we obtain:

$$\begin{aligned} \|v\|_{C_0^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})} \\ \leq C_{\partial+A} \left[(\|\psi\|_{L^s(\tau, T_\bullet)} + C_\alpha(\tau, T_\bullet)) (\|v\|_{C^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})} + \|w\|_{C^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})}) \right. \\ \left. + (1 + \xi^\bullet \mu^\bullet) \|w\|_{\mathbb{W}^{1,s}(\tau, T_\bullet, (W_D^{-1,p}, W_D^{1,p}))} \right]. \end{aligned}$$

We now choose τ to be

$$\tau := \inf \left\{ s \in (0, T^\bullet) : C_{\partial+A} (\|\psi\|_{L^s(s, T^\bullet)} + C_\alpha(s, T^\bullet)) \leq \frac{1}{2} \right\}.$$

Note that $\tau < T^\bullet$. In particular, with the chosen τ , we have

$$C_{\partial+A} (\|\psi\|_{L^s(\tau, T_\bullet)} + C_\alpha(\tau, T_\bullet)) \leq \frac{1}{2}$$

for all $T_\bullet \in (\tau, T^\bullet]$. Thus we are able to absorb $\|v\|_{C_0^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})}$ on the left in the last estimate. Hence, for any $T_\bullet \in (\tau, T^\bullet)$

$$\begin{aligned} \|v\|_{C_0^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})} \leq 2C_{\partial+A} \left[(\|\psi\|_{L^s(0, T)} + C_\alpha(0, T)) \|w\|_{C^{0,\gamma}([\tau, T] \times \bar{\Omega})} \right. \\ \left. + (1 + \xi^\bullet \mu^\bullet) \|w\|_{\mathbb{W}^{1,s}(0, T, (W_D^{-1,p}, W_D^{1,p}))} \right]. \end{aligned}$$

- 4 The right-hand side is *independent* of $T_\bullet$. Thus, denoting the bound by D ,

$$\limsup_{T_\bullet \nearrow T^\bullet} \|v\|_{C^{0,\gamma}([\tau, T_\bullet] \times \bar{\Omega})} \leq D < \infty. \quad (9)$$

This implies that $v \in C([\tau, T^\bullet] \times \bar{\Omega})$: Let $t_k \nearrow T^\bullet$. Then $(v(t_k))$ is bounded in $C^{0,\gamma}(\bar{\Omega})$ by (9) and due to Arzelà-Ascoli, there exists a subsequence (with the same name) such that $v(t_k) \rightarrow V$ in $C(\bar{\Omega})$. Set $v(T^\bullet) := V$. Let $s_k \nearrow T^\bullet$ be arbitrary. By (9), we find

$$\begin{aligned} \|v(T^\bullet) - v(s_k)\|_{C(\bar{\Omega})} &\leq \|v(T^\bullet) - v(t_k)\|_{C(\bar{\Omega})} + \|v(t_k) - v(s_k)\|_{C(\bar{\Omega})} \\ &\leq \|v(T^\bullet) - v(t_k)\|_{C(\bar{\Omega})} + D|t_k - s_k|^\gamma \longrightarrow 0, \end{aligned}$$

- 5 so v is continuous at $T^\bullet$ and $v \in C([\tau, T^\bullet] \times \bar{\Omega})$.

1 Now finally consider the equation

$$\partial z - \nabla \cdot \xi(v + w)\mu \nabla z + z + \mathcal{B}_\alpha z = \mathcal{F}(v + w), \quad z(\tau) = u(\tau). \quad (10)$$

2 Note that $t \mapsto \partial - \nabla \cdot \xi(v(t) + w(t))\mu \nabla + 1$ is continuous on $[\tau, T^\bullet]$ as an oper-
 3 ator from the maximal regularity space into $L^s(\tau, T^\bullet, W_D^{-1,p})$. Thus, $-\nabla \cdot \xi(v +$
 4 $w)\mu \nabla + 1$ satisfies nonautonomous maximal parabolic regularity on $L^s(\tau, T^\bullet, W_D^{-1,p})$
 5 with the domain $W_D^{1,p}$ via Propositions 2.6 and 2.8 due to (6) and the perma-
 6 nence principle in Lemma 2.12. It follows that (10) admits a unique solution
 7 $z \in \mathbb{W}^{1,s}(\tau, T^\bullet, (W_D^{-1,p}, W_D^{1,p}))$. In particular, the limit $\lim_{t \nearrow T^\bullet} z(t) = z(T^\bullet)$ exists
 8 in $(W_D^{-1,p}, W_D^{1,p})_{1/s', s}$ by Lemma 2.7. But by construction $v + w = u$ and thus
 9 $z = u$ on $[\tau, T^\bullet]$, so the blowup criterion (7) must have been false. Thus u is a
 10 global solution. $\square$

11 Note that in the foregoing proof (10) is well defined, although $\mathcal{F}$ is defined only
 12 on $(W_D^{-1,p}, W_D^{1,p})_{1/s', s}$, instead of $C(\bar{\Omega})$ as in [43]. This is because the right hand
 13 side of (10) is still well-defined in $L^s(\tau, T^\bullet; W_D^{-1,p})$ by the growth condition in
 14 Assumption 2.13 ($\mathcal{F}$) and (9).

15 **3.2. First results on improved regularity.** From the previous existence result,
 16 one can hope to bootstrap the regularity of u provided that the data in the problem
 17 is compatible. We show how such an argument could be made. To this end, let us
 18 first cite the following result, adapted to our setting:

Theorem 3.2 ([9], Theorem 3.20). *Let u be the unique solution to (5) from The-
 orem 3.1. Let $\zeta \in (\frac{d}{p}, 1)$ and suppose that $\frac{2}{s} < (\zeta - \frac{d}{p}) \wedge (1 - \zeta)$. Assume further
 that*

$$u_0 \in (H_D^{-\zeta, p}, \text{Dom}_{H_D^{-\zeta, p}}(-\nabla \cdot \mu \nabla + 1))_{1/s', s} \quad \text{and that} \quad \mathcal{F}(u) \in L^s(I, H_D^{-\zeta, p}).$$

19 *Then in fact $u \in \mathbb{W}^{1,s}(I, (H_D^{-\zeta, p}, \text{Dom}_{H_D^{-\zeta, p}}(-\nabla \cdot \mu \nabla + 1)))$ and u is the unique*
 20 *solution to (5) on $H_D^{-\zeta, p}$.*

Let us briefly recall the main idea of the proof in [9], starting from Theorem 3.1.
 The core of the reasoning is that the degree of Hölder continuity of the given solu-
 tion u suffices to show that the operator $-\nabla \cdot \xi(u)\mu \nabla + 1$ satisfies nonautonomous
 maximal parabolic regularity in $H_D^{-\zeta, p}$ when s is large enough, depending on ζ .
 This is established in a quite involved and technical chain of arguments basing on
 the Acquistapace-Terreni condition. From there, one argues quite easily that the
 problem

$$\partial w - \nabla \cdot \xi(u)\mu \nabla w + w = \mathcal{F}(u) - \mathcal{B}_\alpha u \quad \text{in } L^s(I, H_D^{-\zeta, p}), \quad w(0) = u_0$$

21 admits a unique solution $w \in \mathbb{W}^{1,s}(I, (H_D^{-\zeta, p}, \text{Dom}_{H_D^{-\zeta, p}}(-\nabla \cdot \mu \nabla + 1)))$, which, due
 22 to uniqueness of u in the $W_D^{-1,p}$ setting, must coincide with u .

23 We have already seen in Lemma 2.3 that in fact $H_D^{-\zeta, p} = [W_D^{-1,p}, L^p]_{1-\zeta}$. In the
 24 present paper we systematically follow the above bootstrapping ansatz to discuss
 25 solutions to (5) in $[L^p, W_D^{-1,p}]_\theta$ with general $\theta \in [0, 1]$. It is a welcome byproduct
 26 that our reasoning will be less involved than the one in [9] and that we can dispose
 27 of the requirement $\frac{2}{s} < 1 - \theta$.

28 Let us conclude this section by sketching an associated result on the interpolation
 29 scale $[L^{p/2}, W^{-1,p}]_\theta$. Some might see this as the more natural scale as it is well
 30 known that thresholds such as $p > d$ to obtain spatially continuous solutions apply

1 exactly to the two spaces there. Indeed, in this case, an involved bootstrapping
 2 argument is not necessary at all, since improved regularity related to $X = L^{p/2}$ is
 3 an immediate consequence of Theorem 3.2 and $u \in C(\bar{I}, W_D^{1,p})$:

Theorem 3.3. *Consider the setting of Theorem 3.2. Suppose that $\alpha = 0$. Assume additionally that*

$$u_0 \in (L^{p/2}, \text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1))_{1/s', s} \quad \text{and that} \quad \mathcal{F}(u) \in L^s(I, L^{p/2}).$$

4 Then $u \in \mathbb{W}^{1,s}(I, (L^{p/2}, \text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1)))$ and u is the unique solution to (5)
 5 on $L^{p/2}$.

Proof. We bootstrap the solution from Theorem 3.2 once more. (Note that $L^{p/2} \hookrightarrow H_D^{-\zeta, p}$ due to the assumptions on ζ and p and (adjoint) Sobolev embedding.) From Theorem 3.2 and maximal regularity space embeddings, we in fact have $u \in C(\bar{I}, W_D^{1,p})$, see [9, Corollary 3.7] or [25, Lemma 6.16]. But $\text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1)$ is invariant under a coefficient perturbation by a $W_D^{1,p}$ -function, see [25, Lemma 6.7/Corollary 6.8]. Thus

$$\text{Dom}_{L^{p/2}}(-\nabla \cdot \xi(u(t))\mu \nabla + 1) = \text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1) \quad \text{for every } t \in \bar{I},$$

6 in particular

$$t \mapsto -\nabla \cdot \xi(u(t))\mu \nabla + 1 \in C(\bar{I}, \mathcal{L}(\text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1), L^{p/2})).$$

7 Since for each fixed $\tau \in \bar{I}$ the autonomous operator $-\nabla \cdot \xi(u(\tau))\mu \nabla + 1$ admits
 8 maximal parabolic regularity on $L^{p/2}$ (Proposition 2.8) with the common domain
 9 $\text{Dom}_{L^{p/2}}(-\nabla \cdot \mu \nabla + 1)$, we conclude by Proposition 2.6 that also $-\nabla \cdot \xi(u)\mu \nabla + 1$
 10 admits *nonautonomous* maximal parabolic regularity on $L^{p/2}$. The claim follows.
 11 $\square$

12 Let us briefly come back to the motivation outlined in the Introduction, regard-
 13 ing optimality conditions for an optimal control problem associated with (5) with
 14 pointwise constraints on the gradient of the state. For these, we want to guarantee
 15 that $\nabla u \in C(\bar{Q}, \mathbb{R}^d)$ for a solution u to (5). But, in general, we will have $\frac{p}{2} < d$ —see
 16 Remark 2.14—, so this regularity cannot be obtained even in the optimal case for
 17 Theorem 3.3, which is $u \in W^{1,s}(I, L^{p/2}) \cap L^s(I, W^{2,p/2} \cap W_D^{1,p/2})$. In this sense,
 18 even Theorem 3.3 is insufficient for this endeavor. On the scale $X_\theta = [W^{-1,p}, L^p]_\theta$,
 19 however, it is possible to obtain the necessary regularity under quite optimal con-
 20 ditions. This is the main reason not to be content with Theorem 3.3 but to work
 21 on the scale X_θ .

22 **4. Nonautonomous maximal parabolic regularity on the scale.** As outlined
 23 before, the main driving force behind the regularity bootstrapping procedure will be
 24 nonautonomous maximal parabolic regularity for $-\nabla \cdot \xi(u)\mu \nabla + 1$ in better spaces
 25 than $W_D^{-1,p}$. This is the topic of the present section. We formally introduce the
 26 following spaces that will also be used throughout the rest of the paper:

$$X_\theta := [W_D^{-1,p}, L^p]_\theta, \quad Y_\theta := \text{Dom}_{X_\theta}(-\nabla \cdot \mu \nabla + 1), \quad \theta \in (0, 1).$$

27 To simplify notation, we set $X_0 = W_D^{-1,p}$ and $X_1 = L^p$ as well as $Y_0 = W_D^{1,p}$ and
 28 $Y_1 = \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)$. Recall from Lemma 2.3 that for $\theta \neq 1/p$, the space
 29 X_θ in fact coincides with a (dual) Bessel space which either reflects the Dirichlet
 30 boundary condition ($\theta < 1/p$) or not ($\theta > 1/p$). In particular, we have $Y_\theta \hookrightarrow X_\theta$
 31 with dense embedding.

We will be concerned with deriving sharp conditions on the scalar coefficient perturbation η to have maximal parabolic regularity for $-\nabla \cdot \eta \mu \nabla + 1$ in the following problem:

$$\left. \begin{aligned} \partial_t w - \nabla \cdot \eta \mu \nabla w + w &= f && \text{in } L^r(I, X_\theta), \\ w(0) &= w_0 && \text{in } (X_\theta, Y_\theta)_{1/r', r} \end{aligned} \right\} \quad (11)$$

1 Hereby, we expect the coefficient function $\eta: Q \rightarrow \mathbb{R}$ to require a certain Hölder-
 2 regularity that will be made precise subsequently. This is an educated guess based
 3 on the fact that for $\theta = 0$, so $X_\theta = W_D^{-1,p}$, we have seen in Lemma 2.12 and
 4 Propositions 2.6 and 2.8 that $\eta \in C(\bar{Q})$ uniformly continuous is an appropriate
 5 choice.

6 Accordingly, the first and main step is to show an analogue to Lemma 2.12,
 7 that is, that the domain of the elliptic operator $-\nabla \cdot \eta \mu \nabla + 1$ in X_θ is again
 8 $Y_\theta = \text{Dom}_{X_\theta}(-\nabla \cdot \mu \nabla + 1)$ for a positive function $\eta \in C^{0,\vartheta}(\bar{\Omega})$ with $\vartheta \in (\theta, 1]$.

9 To do so we consider the linear map

$$\eta \mapsto (-\nabla \cdot \eta \mu \nabla)(-\nabla \cdot \mu \nabla + 1)^{-1} \quad (12)$$

in the following limiting settings:

$$C(\bar{\Omega}) \rightarrow \mathcal{L}(W_D^{-1,p}) \quad \text{and} \quad C^{0,1}(\bar{\Omega}) \rightarrow \mathcal{L}(L^p).$$

10 These maps are well-defined and continuous; for the first one we refer to Assump-
 11 tion 2.13 (Iso) and Hölder's inequality. The second is taken care of by the following
 12 lemma and $C^{0,1}(\bar{\Omega}) \hookrightarrow W^{1,\infty}(\Omega)$ ([27, Remark 4.2]):

Lemma 4.1. *Let $\eta \in W^{1,\infty}(\Omega)$. Then $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \hookrightarrow \text{Dom}_{L^p}(-\nabla \cdot \eta \mu \nabla)$
 and*

$$\|(-\nabla \cdot \eta \mu \nabla)(-\nabla \cdot \mu \nabla + 1)^{-1}\|_{\mathcal{L}(L^p)} \lesssim \|\eta\|_{W^{1,\infty}}.$$

Proof. Let $u \in \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \hookrightarrow W_D^{-1,p}$ (cf. Proposition 2.10), and let $v \in C_D^\infty(\Omega)$. Then

$$\langle -\nabla \cdot \eta \mu \nabla u, v \rangle := \int_{\Omega} \eta \mu \nabla u \cdot \nabla v = \int_{\Omega} \mu \nabla u \cdot [\nabla(\eta v) - v \nabla \eta].$$

Hence, with $\eta \in W^{1,\infty}$,

$$\begin{aligned} |\langle -\nabla \cdot \eta \mu \nabla u, v \rangle| &\leq \|u\|_{\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)} \|\eta v\|_{L^{p'}} + \|\mu\|_{L^\infty} \|\nabla u\|_{L^p} \|v\|_{L^{p'}} \|\nabla \eta\|_{L^\infty} \\ &\lesssim \|\eta\|_{W^{1,\infty}} \|u\|_{\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)} \|v\|_{L^{p'}}. \end{aligned}$$

13 Since $C_D^\infty(\Omega)$ is dense in $L^{p'}(\Omega)$, it follows that $-\nabla \cdot \eta \mu \nabla u$ defines a continuous linear
 14 functional on $L^{p'}(\Omega)$. So $u \in \text{Dom}_{L^p}(-\nabla \cdot \eta \mu \nabla)$, and the last estimate shows that
 15 $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \hookrightarrow \text{Dom}_{L^p}(-\nabla \cdot \eta \mu \nabla)$. Hence (12) gives rise to a continuous
 16 linear operator on $L^p(\Omega)$. The stated norm estimate also follows immediately. $\square$

17 With bilinear interpolation as in Theorem A.1, we then find that the linear
 18 operator in (12) is also well-defined and continuous as an operator

$$[C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta \rightarrow \mathcal{L}([W_D^{-1,p}, L^p]_\theta) = \mathcal{L}(X_\theta) \quad (13)$$

19 for each $\theta \in [0, 1]$. The following invariance principle, which is similar to [25,
 20 Lemmas 6.6, 6.7], relies on this observation; in fact it will serve as cornerstone of
 21 our further analysis.

22 **Lemma 4.2.** *Let $\theta \in [0, 1]$ be given.*

(1) Let $\eta \in [C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta$. Then $Y_\theta \hookrightarrow \text{Dom}_{X_\theta}(-\nabla \cdot \eta \mu \nabla + 1)$ and the map

$$[C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta \rightarrow \mathcal{L}(Y_\theta, X_\theta), \quad \eta \mapsto -\nabla \cdot \eta \mu \nabla$$

is bounded and linear.

(2) Suppose that $0 < \eta, \eta^{-1} \in [C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta$. Then

$$Y_\theta = \text{Dom}_{X_\theta}(-\nabla \cdot \eta \mu \nabla + 1),$$

with equivalent norms.

In particular, the assertions hold true for a positive function $\eta \in C^{0,\vartheta}(\bar{\Omega})$ with $\vartheta \in (\theta, 1]$ if $\theta < 1$ and $\vartheta = 1$ if $\theta = 1$.

Proof. (1) For the first assertion, it will be enough to show that $Y_\theta \hookrightarrow \text{Dom}_{X_\theta}(-\nabla \cdot \eta \mu \nabla)$. For $\varphi \in Y_\theta$ we have

$$-\nabla \cdot \eta \mu \nabla \varphi = \underbrace{[(-\nabla \cdot \eta \mu \nabla)(-\nabla \cdot \mu \nabla + 1)^{-1}]}_{\in \mathcal{L}(X_\theta) \text{ due to (13)}} \underbrace{(-\nabla \cdot \mu \nabla \varphi + \varphi)}_{\in X_\theta} \in X_\theta,$$

so indeed $\varphi \in \text{Dom}_{X_\theta}(-\nabla \cdot \eta \mu \nabla)$. We further use that the operator in (12) is continuous as in (13) with its norm bounded by

$$\|(-\nabla \cdot \eta \mu \nabla)(-\nabla \cdot \mu \nabla + 1)^{-1}\|_{\mathcal{L}(X_\theta)} \lesssim \|\eta\|_{[C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta}.$$

to estimate

$$\begin{aligned} \|-\nabla \cdot \eta \mu \nabla \varphi\|_{X_\theta} &\leq \|(-\nabla \cdot \eta \mu \nabla)(-\nabla \cdot \mu \nabla + 1)^{-1}\|_{\mathcal{L}(X_\theta)} \|-\nabla \cdot \mu \nabla \varphi + \varphi\|_{X_\theta} \\ &\lesssim \|\eta\|_{[C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta} \|\varphi\|_{Y_\theta}. \end{aligned}$$

The assertions follow.

(2) Let $\eta, \eta^{-1} \in [C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta$. Then both η and η^{-1} are uniformly continuous, bounded and positive. With Lemma 2.12 it follows that $-\nabla \cdot \eta \mu \nabla + 1$ is also a topological isomorphism between $W_D^{1,p}$ and $W_D^{-1,p}$, i.e., $\hat{\mu} := \eta \mu$ satisfies the isomorphism assumption in Assumption 2.13 (Iso). Consequently, we can apply the arguments of the first part to $\hat{\eta} := \eta^{-1}$ and $\hat{\mu}$ instead of η and μ , respectively, to obtain that

$$\begin{aligned} \text{Dom}_{X_\theta}(-\nabla \cdot \eta \mu \nabla + 1) &= \text{Dom}_{X_\theta}(-\nabla \cdot \hat{\mu} \nabla + 1) \\ &\hookrightarrow \text{Dom}_{X_\theta}(-\nabla \cdot \hat{\eta} \hat{\mu} \nabla + 1) = \text{Dom}_{X_\theta}(-\nabla \cdot \mu \nabla + 1), \end{aligned}$$

and this implies the claim.

Finally, by Proposition A.2 we have $C^{0,\vartheta}(\bar{\Omega}) \hookrightarrow [C(\bar{\Omega}), C^{0,1}(\bar{\Omega})]_\theta$ when $\vartheta \in (\theta, 1]$. Moreover, if $\eta \in C^{0,\vartheta}(\bar{\Omega})$ is positive, then also $\eta^{-1} \in C^{0,\vartheta}(\bar{\Omega})$. Hence, for such η all the previous assertions are valid. The case $\theta = 1$ follows analogously. $\square$

Next, we establish nonautonomous maximal parabolic regularity of the operator $-\nabla \cdot \eta \mu \nabla + 1$ on X_θ for positive coefficient functions $\eta \in C(\bar{I}, C^{0,\vartheta}(\bar{\Omega}))$ for $\vartheta \in (0, 1]$. This will give existence, uniqueness and regularity of solutions to (11). Since for $\eta \in C(\bar{I}, C^{0,\vartheta}(\bar{\Omega}))$ for $\vartheta \in (0, 1]$ we already have constant domains for $-\nabla \cdot \eta \mu \nabla + 1$ in X_θ at hand by virtue of Lemma 4.2, the following theorem follows quite easily with Proposition 2.6:

Theorem 4.3. *Given $\vartheta \in (0, 1]$ and $\theta \in [0, \vartheta]$ if $\vartheta < 1$ and $\theta \in [0, 1]$ if $\vartheta = 1$, let $\eta \in C(\bar{I}, C^{0,\vartheta}(\bar{\Omega}))$ be positive. Then the operator $-\nabla \cdot \eta \mu \nabla + 1$ exhibits nonautonomous maximal parabolic regularity on X_θ with domain Y_θ , that is, for any $r \in (1, \infty)$, if*

$f \in L^r(I, X_\theta)$ and $w_0 \in (X_\theta, Y_\theta)_{1/r', r}$, then there is a unique solution w to (11) and the corresponding solution map

$$L^r(I, X_\theta) \times (X_\theta, Y_\theta)_{1/r', r} \rightarrow \mathbb{W}^{1, r}(I, (X_\theta, Y_\theta)), \quad (f, w_0) \mapsto w, \quad (14)$$

is a topological isomorphism. If $\mathcal{E} \subset C(\bar{I}, C^{0, \vartheta}(\bar{\Omega}))$ is a compact set such that $\eta(x) > 0$ for all $x \in \bar{Q}$ for all $\eta \in \mathcal{E}$, then the isomorphism (14) is equicontinuous with respect to $\eta \in \mathcal{E}$.

Proof. Due to Lemma 4.2 the map

$$\bar{I} \rightarrow \mathcal{L}(Y_\theta, X_\theta), \quad t \mapsto -\nabla \cdot \eta(t)\mu\nabla + 1,$$

is well-defined and continuous, and for every t , the resulting operator is even continuously invertible. This together with Corollary 2.9 allows to invoke Proposition 2.6 which yields the claim on nonautonomous maximal parabolic regularity on X_θ with the domain Y_θ .

It remains to argue on equicontinuity of the solution maps for $\eta \in \mathcal{E}$: Note that for each $r \in (1, \infty)$ the map

$$C(\bar{I}, C^{0, \vartheta}(\bar{\Omega})) \supset \mathcal{E} \rightarrow \mathcal{L}(\mathbb{W}^{1, r}(I, (X_\theta, Y_\theta)), L^r(I, X_\theta) \times (X_\theta, Y_\theta)_{1/r', r}),$$

$$\eta \mapsto (\partial - \nabla \cdot \eta\mu\nabla + 1, \gamma_0)^{-1},$$

is well-defined due to nonautonomous parabolic regularity of $-\nabla \cdot \eta\mu\nabla + 1$ in X_θ with domain Y_θ for each $\eta \in \mathcal{E}$. Moreover, the map is continuous by Lemma 4.2 and continuity of operator inversion $A \mapsto A^{-1}$. Hence, its operator norm is bounded on the compact set $\mathcal{E}$. $\square$

Remark 4.4. The significant difference between the proof of Theorem 4.3 and the one of [9, Theorem 3.20] is that here, by virtue of Lemma 4.2, the domains of $-\nabla \cdot \eta(t)\mu\nabla + 1$ in X_θ are *a priori* independent of t . (It follows *a posteriori* in the proof of [9, Theorem 3.20], though.) This allows to apply Proposition 2.6 basing on constant domains which is much less involved than the treatment for *a priori* nonconstant domains based on uniform $\mathcal{R}$ -boundedness as in [46]. The present insight that the domains are constant *a priori* is in turn based on the bilinear interpolation technique as established in Appendix A.1 and invariance of Hölder-continuous functions under taking the reciprocal.

5. Abstract quasilinear parabolic equations in the scale. Building upon on nonautonomous maximal parabolic regularity for (11) on X_θ as in Theorem 4.3, we now treat the quasilinear abstract problem (5) on X_θ utilizing a bootstrapping argument. This is the main achievement of the paper. The bootstrapping procedure starts from Theorem 3.1 for $X_0 = W_D^{-1, p}$ and works its way up through X_θ until some maximal $\bar{\theta} \in (0, 1]$ is reached. The fundamental idea is that given the solution u in X_θ , one finds—via Theorem 4.3—that the associated degree of Hölder continuity of u is in fact sufficient to obtain nonautonomous maximal parabolic regularity for $-\nabla \cdot \xi(u)\mu\nabla + 1$ in X_{θ^+} for $\theta^+ > \theta$.

The value of this maximal $\bar{\theta}$ will be related to the degree of Hölder-regularity κ of the elliptic boundary value problem in L^p associated to $-\nabla \cdot \mu\nabla + 1$. In view of Theorem 4.3, this should not come as a surprise, since the highest degree of Hölder-regularity obtainable within the X_θ scale will be limited exactly by the associated value for $X_1 = L^p$.

Thus, we start with a short discussion regarding κ . The main argument is then given for the case where $\mathcal{F}$ in the abstract quasilinear equation (5) is *constant* in order to make the idea more transparent. This is Section 5.2. We then include $\mathcal{F}$ in Section 5.3.

Definition 5.1. Let $\kappa \in [1 - \frac{d}{p}, 2 - \frac{d}{p}]$ be as large as possible such that

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \hookrightarrow \begin{cases} C^{[\kappa], \kappa}(\overline{\Omega}) & \text{if } \kappa \neq 1, \\ C^{0,1}(\overline{\Omega}) & \text{if } \kappa = 1. \end{cases}$$

Let us point out that such a $\kappa \geq 1 - \frac{d}{p}$ always exists under Assumption 2.13 (Iso), since

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \hookrightarrow W_D^{1,p} \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega}).$$

Remark 5.2. We comment on particular cases for κ in Definition 5.1.

- (1) The optimal case $\kappa = 2 - \frac{d}{p}$ is obtained for example when Ω is a $C^{1,1}$ -domain, μ is Lipschitz-continuous, and pure homogeneous Dirichlet or Neumann boundary conditions are imposed, so either $\Gamma_D = \partial\Omega$ or $\Gamma_D = \emptyset$. In this case we have optimal $W^{2,p}$ -regularity for the boundary problem associated to $-\nabla \cdot \mu \nabla + 1$, so $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) = W^{2,p} \cap W_D^{1,p}$, see [21, Theorems 2.4.2.5 and 2.4.2.7].
- (2) Suppose that the domain geometry remains rather irregular, but μ gives rise to a multiplier on $H^{\eta,p}$ for some $\eta < 1/p$, possibly small. Then it can be shown that a fractional power of the L^p -realization of $-\nabla \cdot \mu \nabla + 1$ will have the domain $H_D^{1+\varepsilon,p}$ for some $\varepsilon < \eta$ by extrapolation techniques, see [42, Section 3]. For example, μ could be the characteristic function of a convex subset of Ω , or a subset of locally bounded perimeter, or a linear combination of such functions. We do not go into details here. In any case, it follows that we can have $\kappa = 1 + \varepsilon - \frac{d}{p} > 1 - \frac{d}{p}$ in this quite general situation.

The maximal possible value of κ is nontrivial to determine; it is related to the regularity of the boundary of Ω , the boundary conditions imposed, the coefficient function μ , and p , and may not be known explicitly in general. We thus formulate our further results in dependence of κ . See also Appendix B for a further discussion for various practical situations.

5.1. The interpolation spaces and fractional powers. In this section we collect some results on the interpolation spaces

$$[X_\vartheta, Y_\vartheta]_\tau := \left[[W_D^{-1,p}, L^p]_\vartheta, \text{Dom}_{X_\vartheta}(-\nabla \cdot \mu \nabla + 1) \right]_\tau.$$

These occur naturally in the consideration of quasilinear problems from trace embeddings, see Lemma 2.7; usually, they are considered in order to establish regularity properties for the nonlinear functions in fixed point arguments. In view of the results of Section 4 on nonautonomous maximal parabolic regularity for Hölder-continuous coefficients, it is thus not surprising that we aim for an embedding into a suitable Hölder space for $[X_\vartheta, Y_\vartheta]_\tau$.

The first auxiliary result is that the spaces in question can be identified with the domain of a fractional power of $-\nabla \cdot \mu \nabla + 1$, co-restricted to L^p .

Lemma 5.3. Let $\vartheta \in [0, 1]$ and $\tau \in (\frac{1}{2} - \frac{\vartheta}{2}, 1]$. Then

$$[X_\vartheta, Y_\vartheta]_\tau = \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}}.$$

Proof. Recall that the fractional powers of $-\nabla \cdot \mu \nabla + 1$ were well-defined and that we even have bounded imaginary powers at our disposal (Proposition 2.8). We thus apply Lemma A.4 and $\text{Dom}_{W_D^{-1,p}}(-\nabla \cdot \mu \nabla + 1) = W_D^{1,p}$ by Assumption 2.13 (Iso):

$$[X_\vartheta, Y_\vartheta]_\tau = \left[[W_D^{-1,p}, W_D^{1,p}]_\tau, [L^p, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)]_\tau \right]_{\tau, \vartheta}. \quad (15)$$

Next, we branch along $\tau = \frac{1}{2}$. If $\frac{1}{2} - \frac{\vartheta}{2} < \tau < \frac{1}{2}$, then necessarily $\vartheta > 0$ and we argue as follows:

$$\begin{aligned} & \left[[W_D^{-1,p}, W_D^{1,p}]_\tau, [L^p, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)]_\tau \right]_{\tau, \vartheta} \\ &= \left[[W_D^{-1,p}, W_D^{1,p}]_\tau, [L^p, W_D^{1,p}]_{2\tau} \right]_{\tau, \vartheta} = \left[[W_D^{-1,p}, L^p]_\xi, W_D^{1,p} \right]_{(1+\vartheta)\tau} \end{aligned}$$

where we have used the Kato square root property $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\frac{1}{2}} = W_D^{1,p}$ (Proposition 2.10) and the reiteration theorem for complex interpolation with $(1 - (1 + \vartheta)\tau)\xi = (1 - 2\tau)\vartheta$. Now, $(1 + \vartheta)\tau - (1 - (1 - \vartheta)\tau)(1 - \xi) = 2\tau - 1 + \vartheta > 0$. Hence, using Lemma 2.3 and the reiteration theorem twice,

$$\begin{aligned} \left[[W_D^{-1,p}, L^p]_\xi, W_D^{1,p} \right]_{(1+\vartheta)\tau} &= \left[[W_D^{-1,p}, [W_D^{-1,p}, W_D^{1,p}]_{\frac{1}{2}}]_\xi, W_D^{1,p} \right]_{(1+\vartheta)\tau} \\ &= [W_D^{-1,p}, W_D^{1,p}]_{\tau + \frac{\vartheta}{2}} \\ &= [L^p, W_D^{1,p}]_{2\tau - 1 + \vartheta} = \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}}. \end{aligned}$$

Next, let $\tau \geq \frac{1}{2}$ and $\tau > \frac{1}{2}$ if $\vartheta = 0$. Then, again by reiteration and Lemma 2.3 we have

$$[W_D^{-1,p}, W_D^{1,p}]_\tau = \left[[W_D^{-1,p}, W_D^{1,p}]_{\frac{1}{2}}, W_D^{1,p} \right]_{2\tau - 1} = [L^p, W_D^{1,p}]_{2\tau - 1}.$$

Again, the Kato square root property implies that $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\frac{1}{2}} = W_D^{1,p}$. Hence, by reiteration for fractional power domains (Lemma 2.11) in (15),

$$\begin{aligned} [X_\vartheta, Y_\vartheta]_\tau &= \left[[L^p, W_D^{1,p}]_{2\tau - 1}, [L^p, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)]_\tau \right]_{\tau, \vartheta} \\ &= \left[\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2}}, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^\tau \right]_{\tau, \vartheta} \\ &= \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}}. \end{aligned}$$

¹ This was the claim. □

Lemma 5.4. *Let $\vartheta \in [0, 1]$ and $\tau \in (\frac{1}{2} + \frac{d}{2p} - \frac{\vartheta}{2}, 1]$. Assume that $\kappa \leq 1$ in Definition 5.1. Set*

$$\sigma := 2\tau - 1 + \vartheta - \frac{d}{p} \quad \text{if} \quad \tau < 1 - \frac{\vartheta}{2}$$

and

$$\sigma := 1 - \frac{d}{p} + (2\tau - 2 + \vartheta) \left(\kappa - 1 + \frac{d}{p} \right) \quad \text{if} \quad \tau \geq 1 - \frac{\vartheta}{2}.$$

Then

$$[X_\vartheta, Y_\vartheta]_\tau \hookrightarrow C^{0, \sigma}(\overline{\Omega}).$$

Proof. We argue via Lemma 5.3. Let first $\tau < 1 - \frac{\vartheta}{2}$. Then $\tau - \frac{1}{2} + \frac{\vartheta}{2} < \frac{1}{2}$ and we find with Sobolev embedding (Lemma 2.2)

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}} = H_D^{2\tau - 1 + \vartheta, p} \hookrightarrow C^{0, 2\tau - 1 + \vartheta - \frac{d}{p}} = C^{0, \sigma}(\overline{\Omega}).$$

Now turn to $\tau \geq 1 - \frac{\vartheta}{2}$. Then $\tau - \frac{1}{2} + \frac{\vartheta}{2} \geq \frac{1}{2}$, hence,

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}} = \left[\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\frac{1}{2}}, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \right]_{2\tau - 2 + \vartheta}$$

and so, via Definition 5.1 for κ , the assumption $\kappa \leq 1$ and Proposition A.2,

$$\begin{aligned} \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}} &= \left[W_D^{1, p}, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \right]_{2\tau - 2 + \vartheta} \\ &\hookrightarrow \left[C^{0, 1 - \frac{d}{p}}(\overline{\Omega}), C^{0, \kappa}(\overline{\Omega}) \right]_{2\tau - 2 + \vartheta} \\ &\hookrightarrow \left(C^{0, 1 - \frac{d}{p}}(\overline{\Omega}), C^{0, \kappa}(\overline{\Omega}) \right)_{2\tau - 2 + \vartheta, \infty} = C^{0, \sigma}(\overline{\Omega}). \quad \square \end{aligned} \quad (16)$$

Unfortunately we did not succeed in proving an analogue to Lemma 5.4 for the case $\kappa > 1$ under our general assumptions. As one sees from (16), with our technique the question boils down to an interpolation result of the form

$$\left(C^{0, 1 - \frac{d}{p}}(\overline{\Omega}), C^{1, \kappa - 1}(\overline{\Omega}) \right)_{\xi, \infty} \hookrightarrow C^{0, 1}(\overline{\Omega}) \quad \text{when} \quad (1 - \xi)(1 - \frac{d}{p}) + \xi\kappa > 1.$$

- 1 We conjecture that such a result will be true for our case of a weak Lipschitz domain.
- 2 We can indeed prove it for example when Ω admits a C^1 -boundary by extension
- 3 techniques, see [38, Example 1.1.9]. This is also in line with the rule of thumb
- 4 that $\kappa > 1$ in Definition 5.1 will require a more smooth situation than we generally
- 5 suppose within this work. For now we content ourselves with making the following

Assumption 5.5. If $\kappa > 1$ in Definition 5.1, we assume that we have for $\vartheta \in [0, 1]$ and $\tau \in [1 - \frac{\vartheta}{2}, 1]$,

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}} \hookrightarrow C^{0, \sigma}(\overline{\Omega})$$

with

$$\sigma := 1 \wedge \left(1 - \frac{d}{p} + (2\tau - 2 + \vartheta)(\kappa - 1 + \frac{d}{p}) \right).$$

Remark 5.6. If in fact the optimal regularity

$$\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) = W^{2, p} \cap W_D^{1, p}$$

is valid, then we can circumvent the Hölder interpolation problem and Assumption 5.5 is indeed true. In this case, $\kappa = 2 - \frac{d}{p}$ by Sobolev embedding, so in (16) we obtain

$$\begin{aligned} \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)^{\tau - \frac{1}{2} + \frac{\vartheta}{2}} &= \left[W_D^{1, p}, \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) \right]_{2\tau - 2 + \vartheta} \\ &= H^{2\tau + \vartheta - 1, p} \cap W_D^{1, p} \hookrightarrow C^{0, \sigma}(\overline{\Omega}). \end{aligned}$$

- 6 (See the extension arguments in Remark B.1 together with [6, Lemma 3.1 and Corol-
- 7 lary 3.7] to justify interpolation and Sobolev embeddings here.) Thus, Assump-
- 8 tion 5.5 in fact only covers the quite particular case where we do not have optimal
- 9 Sobolev regularity for $-\nabla \cdot \mu \nabla + 1$ in L^p , but still $\kappa > 1$ in Definition 5.1.

5.2. Regularity bootstrapping: simple case. We now present the bootstrapping procedure to obtain *global* solutions of (5) on the X_θ -scale. To make the ideas more transparent, we first assume that $\alpha \equiv 0$ and that $\mathcal{F} \equiv f$ is constant, resulting in the following problem:

$$\begin{cases} \partial_t u - \nabla \cdot \xi(u) \mu \nabla u + u = f & \text{in } L^s(I, X_\theta), \\ u(0) = u_0 & \text{on } (X_\theta, Y_\theta)_{1/s', s}. \end{cases} \quad (17)$$

Note that this problem is still quasilinear and therefore shares major characteristic difficulties with (5). We will come back to the full equation (5) in the subsequent section. The deciding factor on the size of θ that we can afford is the number

$$\bar{\theta} := \frac{1 - \frac{d}{p}}{2 - \frac{d}{p} - \kappa}.$$

- 1 Note that $\bar{\theta} \geq 1 - \frac{d}{p}$; moreover, $\bar{\theta} \geq 1$, that is, we can reach $X_1 = L^p$, if and only if
- 2 $\kappa \geq 1$. With this, the main regularity result for (17), and thereby the first version
- 3 of the main theorem, reads as follows:

Theorem 5.7. *Adopt Assumption 5.5 if $\kappa > 1$ and fix $\theta \in [0, \bar{\theta}) \cap [0, 1]$. Suppose that*

$$\frac{1}{s} < \frac{1}{2} - \frac{d}{2p} \quad \text{if } \theta < 1 - \frac{d}{p} \quad \text{and} \quad \frac{1}{s} < \frac{1 - \frac{d}{p} - \theta}{2(\kappa - 1 + \frac{d}{p})} + \frac{\theta}{2} \quad \text{otherwise.}$$

- 4 *If $f \in L^s(I, X_\theta)$ and $u_0 \in (X_\theta, Y_\theta)_{1/s', s}$, then the unique solution $u \in \mathbb{W}^{1,s}(I, (X_0, Y_0))$*
- 5 *of (17) on X_0 in fact satisfies $u \in \mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))$, in particular, $u \in C^\beta(\bar{I}, C^{0,\theta}(\bar{\Omega}))$,*
- 6 *and it is the unique solution to (17) on X_θ . Moreover, the solution map $(u_0, f) \mapsto u$*
- 7 *associated with (17) maps bounded sets in $(X_\theta, Y_\theta)_{1/s', s} \times L^s(I, X_\theta)$ into bounded*
- 8 *sets in $\mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))$.*

- 9 To make the structure of the proof of Theorem 5.7 more transparent, we first
- 10 state the required Hölder embedding as an auxiliary result. Of course we use κ
- 11 from Definition 5.1. Note that the second case in the following lemma is void if
- 12 $\kappa = 1 - \frac{d}{p}$.

Lemma 5.8. *Suppose that Assumption 5.5 is valid if $\kappa > 1$. For $\vartheta \in [0, 1]$, let either*

$$0 \leq \sigma < 1 - \frac{d}{p} \quad \text{and} \quad \frac{1}{s} < \frac{1 - \frac{d}{p} - \sigma}{2} + \frac{\vartheta}{2}.$$

or

$$\sigma \in \left[1 - \frac{d}{p}, 1 - \frac{d}{p} + \vartheta\left(\kappa - 1 + \frac{d}{p}\right)\right) \cap [0, 1] \quad \text{and} \quad \frac{1}{s} < \frac{1 - \frac{d}{p} - \sigma}{2(\kappa - 1 + \frac{d}{p})} + \frac{\vartheta}{2}.$$

- 13 *Then $\mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta)) \hookrightarrow_c C(\bar{I}, C^{0,\sigma}(\bar{\Omega}))$.*

Proof. By Bochner-Sobolev embeddings, see Lemma 2.7, we get

$$\mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta)) \hookrightarrow_c C(\bar{I}, [X_\vartheta, Y_\vartheta]_\tau), \quad (18)$$

as long as $0 \leq \tau < 1 - \frac{1}{s}$. (Note that $Y_\vartheta \hookrightarrow W_D^{1,p} \hookrightarrow_c L^p \hookrightarrow X_\vartheta$.) Thus, the claim follows via Lemma 5.4 in the following cases:

$$\sigma < 1 - \frac{d}{p} \quad \text{and} \quad \tau := \frac{1}{2} + \frac{d}{2p} - \frac{\vartheta}{2} + \frac{\sigma}{2},$$

or (here we use Assumption 5.5 when $\kappa > 1$)

$$\sigma \in \left[1 - \frac{d}{p}, 1 - \frac{d}{p} + \vartheta\left(\kappa - 1 + \frac{d}{p}\right)\right] \cap [0, 1] \quad \text{and} \quad \tau := 1 - \frac{\vartheta}{2} + \frac{\sigma - 1 + \frac{d}{p}}{2(\kappa - 1 + \frac{d}{p})}.$$

1 The conditions on s in the statement correspond exactly to $\frac{1}{s} < 1 - \tau$, so $\tau <$
 2 $1 - \frac{1}{s}$, for each case, hence the choices of τ and s are compatible and we obtain the
 3 assertion. $\square$

4 We now come back to proving Theorem 5.7.

5 *Proof of Theorem 5.7.* We start with $\vartheta = 0$. With Theorem 3.1 there exists a
 6 unique solution $u \in \mathbb{W}^{1,s}(I, (X_0, Y_0))$ to (17) on X_0 . Now fix $\epsilon > 0$ such that

$$\frac{1}{s} + \frac{\epsilon}{2} < \frac{1}{2} - \frac{d}{2p} \quad \text{if} \quad \theta < 1 - \frac{d}{p} \quad (19)$$

7 and

$$\frac{1}{s} + \frac{\epsilon}{2(\kappa - 1 + \frac{d}{p})} < \frac{1 - \frac{d}{p} - \theta}{2(\kappa - 1 + \frac{d}{p})} + \frac{\theta}{2} \quad \text{if} \quad \theta \geq 1 - \frac{d}{p}. \quad (20)$$

8 We proceed iteratively as follows:

(1) Suppose that $u \in \mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta))$ for some $\vartheta \in [0, \theta]$. By Lemma 5.8, we have
 $u \in C(\bar{I}, C^{0,\sigma}(\bar{\Omega}))$ for $\sigma := \vartheta + \epsilon$ by the choices of ϵ in (19)/(20) and of s . Note
 here that for $\theta \geq 1 - \frac{d}{p}$, with the choice of ϵ as in (20), we have

$$\frac{1}{s} + \frac{\epsilon}{2} \leq \frac{1}{s} + \frac{\epsilon}{2(\kappa - 1 + \frac{d}{p})} < \frac{1 - \frac{d}{p} - \theta}{2(\kappa - 1 + \frac{d}{p})} + \frac{\theta}{2} \leq \frac{1}{2} - \frac{d}{2p},$$

9 so ϵ is also valid for (19).

10 (2) Set $\vartheta^+ := \min(\vartheta + \frac{\epsilon}{2}, \theta)$. Then Theorem 4.3 with $\eta = \xi(u)$ yields nonau-
 11 tonomous maximal parabolic regularity of $-\nabla \cdot \xi(u)\mu\nabla + 1$ on X_{ϑ^+} . Since
 12 $u_0 \in (X_\vartheta, Y_\vartheta)_{1/s', s} \hookrightarrow (X_{\vartheta^+}, Y_{\vartheta^+})_{1/s', s}$ and $f \in L^s(I, X_\vartheta) \hookrightarrow L^s(I, X_{\vartheta^+})$ by
 13 assumption, we conclude that indeed $u \in \mathbb{W}^{1,s}(I, (X_{\vartheta^+}, Y_{\vartheta^+}))$ is the unique
 14 solution to (17) in X_{ϑ^+} .

15 (3) Note that ϵ is independent of the current ϑ , so we can repeat the foregoing
 16 procedure a finite number of times until $\vartheta^+ = \theta$ is reached.

17 The Hölder regularity stated for u follows immediately from the construction, see
 18 also Lemma 5.8 and the choices of s and θ .

19 Finally, we argue on the statement on mapping bounded sets into bounded sets.
 20 Such a result is already shown in [43, Corollary 5.8] for $\vartheta = 0$. Furthermore, the
 21 embeddings of $\mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta))$ into $C(\bar{I}, C^{0,\sigma}(\bar{\Omega}))$ which were utilized along the
 22 way are *all* compact, cf. Lemma 5.8. Thus, the uniformity statement in Theorem 4.3
 23 is carried through every step of the bootstrapping argument. $\square$

24 The above theorem may be regarded as an extension of [9, Theorem 3.20]:

25 **Remark 5.9.** Let $\kappa = 1 - \frac{d}{p}$. Then we have $\theta < \bar{\theta} = 1 - \frac{d}{p}$. Theorem 5.7 yields
 26 optimal regularity solutions of (5) on

$$X_\theta = [W_D^{-1,p}, L^p]_\theta = H_D^{-\zeta, p}$$

27 with $-\zeta = \theta - 1$, i.e., $\zeta = 1 - \theta > 1 - (1 - \frac{d}{p}) = \frac{d}{p}$. This yields the same assumption on
 28 ζ as in [9]. (Due to $p \leq d+1$, we also have $\theta = 1 - \zeta < 1/p$ in this case.) For the same
 29 value of ζ , we do however require less integrability for s since with our technique

we do not have to pass through $u(t) \in W_D^{1,p}$ as in [9, Proof of Thm. 3.20, Step 3]. Therefore, without imposing *any* new assumption on κ , Theorem 5.7 reproduces and improves the result from [9, Theorem 3.20] which was obtained there by much more involved reasoning.

5.3. Regularity bootstrapping: general case. Essentially the same bootstrapping technique as the one used in the proof of Theorem 5.7 also yields unique solutions of the full problem (5) in the X_θ -scale. This is the second version of our main result.

Theorem 5.10. *Let θ and s be as in Theorem 5.7 and let $u_0 \in (X_\theta, Y_\theta)_{1/s', s}$. If $\theta \geq \frac{1}{p}$, suppose that $\alpha \equiv 0$. Assume that there is $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and all $\vartheta \in [0, \theta]$ the implication*

$$\left[w \in \mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta)) \implies \mathcal{F}(w) \in L^s(I, X_{\min(\theta, \vartheta + \frac{\epsilon}{2})}) \right] \quad (21)$$

holds true. Then, the unique solution $u \in \mathbb{W}^{1,s}(I, (X_0, Y_0))$ of (5) on X_0 in fact satisfies $u \in \mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))$, in particular, $u \in C^\beta(\bar{I}, C^{0,\theta}(\bar{\Omega}))$, and it is the unique solution to (5) on X_θ .

Proof. Since the main line of the proof is the same as for Theorem 5.7, we only comment on the changes: The fixed right hand side f in (17) is replaced by

$$g := \mathcal{F}(u) - \mathcal{B}_\alpha u.$$

Consequently, we have to make sure that if $u \in \mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta))$, then $g \in L^s(I, X_{\vartheta+})$ in order to go through step (ii) of the bootstrapping argument as before. For $\mathcal{F}(u)$, this is ensured exactly by the assumed property (21), while we refer to Lemma 2.5 for $\mathcal{B}_\alpha u$; recall (6). The latter term $\mathcal{B}_\alpha u$ is void if $\theta \geq \frac{1}{p}$. $\square$

Corollary 5.11. *Adopt the assumptions of Theorem 5.10, but let $\alpha \equiv 0$ also when $\theta < \frac{1}{p}$. Denote by $\mathcal{G}$ the family of nonlinear functions $\mathcal{F}$ satisfying the standing Assumption 2.13 and assumption (21), and for which in addition there is a constant $C_{\mathcal{G}} > 0$ for which*

$$\sup_{w \in \mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))} \|\mathcal{F}(w)\|_{L^s(I, X_\theta)} \leq C_{\mathcal{G}}.$$

Denote by $u_{\mathcal{F}}$ the unique solution $u \in \mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))$ to (5) supplied by Theorem 5.10. Then, the set $\{u_{\mathcal{F}} : \mathcal{F} \in \mathcal{G}\}$ is bounded in $\mathbb{W}^{1,s}(I, (X_\theta, Y_\theta))$.

Proof. This follows directly from the statement about the solution operator mapping bounded sets into bounded sets in Theorem 5.7, with $f = \mathcal{F}(u)$. $\square$

Remark 5.12. We mention a few examples for nonlinearities fulfilling the assumption in Theorem 5.10:

- (1) Any nonlinear function given by a Nemytskii operator induced by a sufficiently regular real function with suitable growth bounds—or a bounded one—and also suitably monotone ones will do the trick. Analogous boundary terms can be treated only for $\theta < \frac{1}{p}$.
- (2) Similarly, the nonlocal term from Remark 2.15 obviously fulfills (21).
- (3) A drift-type term as in Remark 2.15 is also covered by the assumptions, if the following additional (sufficient) conditions are fulfilled:

$$g \in L^s(I, H^{1+\theta, p}), \quad \text{and} \quad \mu \in C^{0, \min(1, \tau)}(\bar{\Omega})^{d \times d} \text{ for some } \tau > \theta.$$

Let us verify (21). We are quite concise and borrow several arguments from [42, Section 3]. Suppose first that $\vartheta + \frac{\varepsilon}{2} \leq \theta$. For almost every $t \in I$, we have $\nabla g(t) \in H^{\theta,p} \hookrightarrow H^{\vartheta+\frac{\varepsilon}{2},p}$. On the other hand, from $w \in \mathbb{W}^{1,s}(I, (X_\vartheta, Y_\vartheta))$ it follows that $w \in C(\bar{I}, C^{0,\vartheta+\varepsilon})$ by Lemma 5.4 as in the proof of Theorem 5.10. But then $w(t)\mu$ is Hölder continuous of degree $> \vartheta + \frac{\varepsilon}{2}$ and thus a multiplier on $H^{\vartheta+\frac{\varepsilon}{2},p}$ for every $t \in \bar{I}$. Therefore,

$$\begin{aligned} \|\nabla \cdot w\mu \nabla g\|_{L^s(I, X_{\vartheta+\frac{\varepsilon}{2}})} &\leq \|w\mu \nabla g\|_{L^s(I, H^{\vartheta+\frac{\varepsilon}{2},p})} \\ &\leq \sup_{t \in \bar{I}} \|w(t)\mu\|_{\mathcal{L}(H^{\vartheta+\frac{\varepsilon}{2},p})} \|g\|_{L^s(I, H^{1+\vartheta+\frac{\varepsilon}{2},p})} < \infty. \end{aligned}$$

1 The analogous argument applies when $\theta \geq \vartheta + \frac{\varepsilon}{2}$. (The case $\theta = 1$ is special
2 but straightforward since Lipschitz functions are multipliers on $H^{1,p}$.)

3 **Remark 5.13.** We comment on the optimal case $\text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1) = W^{2,p} \cap$
4 $W_D^{1,p}$.

- (1) It was already mentioned in Remark 5.2 that in the optimal case, $\kappa = 2 - d/p$. Also, we can identify Y_θ in a more explicit manner. Indeed, as in Remark 5.6 we have

$$Y_\theta = [W_D^{1,p}, W^{2,p} \cap W_D^{1,p}]_\theta = H^{1+\theta,p} \cap W_D^{1,p}.$$

5 Thus, Y_θ coincides with $H_D^{1+\theta,p}$ if $\theta < 1/p$. Moreover, recall that for $\theta \neq 1/p$,
6 Lemma 2.3 tells us that X_θ is given precisely by $H_D^{\theta-1,p}$ and $H^{\theta-1,p}$ when
7 $\theta < 1/p$ and $\theta > 1/p$, respectively. In the case $\theta = 1$, the condition on s
8 in Theorems 5.7 and 5.10 becomes $\frac{1}{s} < \frac{1}{2} - \frac{d}{2p}$ which was the one posed in
9 Assumption 2.1, and which is also required for the starting point giving a global
10 solution in the first place, Theorem 3.1. In this sense, it is optimal.

- 11 (2) In the optimal case, Theorem 5.10 with $\theta = 1$ essentially reproduces [12, The-
12 orem 2.3]. In fact, we can even avoid the monotonicity assumption there and
13 we do not have to require that the input data is $2s$ -integrable in time to ob-
14 tain an s -integrable solution. We concede that the assumption on s is weaker
15 in [12], though; this is because the authors there do not pass through θ -Hölder
16 coefficients, and, thus, solutions, but only require *continuous* ones. Note that
17 there seems to be a mistake in the assumptions on time integrability in [12,
18 Theorem 2.3] as we have validated with the authors. In our present notation,
19 the condition there should be $\frac{1}{s} < 1 - \frac{d}{2p}$ which is still less strict than ours.
20 However, as mentioned above, for the optimal $\kappa = 2 - d/p$ and $\theta = 1$, our
21 requirement on s falls back to the one required for Theorem 3.1, the starting
22 point for the whole procedure. In this sense, it is optimal for our technique.

23 **Appendix A. Results related to interpolation theory.** In the following we
24 collect some auxiliary results related to interpolation theory that are required in
25 Sections 4 and 5.

A.1. Bilinear interpolation. Let A, B, X, Y be Banach spaces with continuous and dense inclusions $A \hookrightarrow B$ and $X \hookrightarrow Y$. Further, let bounded linear maps $\Phi: A \rightarrow \mathcal{L}(X)$, $\Psi: B \rightarrow \mathcal{L}(Y)$ be given such that the following compatibility property holds:

$$\text{for all } a \in A: \quad \Psi(a)|_X = \Phi(a). \quad (22)$$

Theorem A.1. *Adopt the above setting and (22). Then, for every $\theta \in [0, 1]$, the restriction of Ψ to $[A, B]_\theta$ gives rise to a bounded linear operator*

$$\Psi: [A, B]_\theta \rightarrow \mathcal{L}([X, Y]_\theta).$$

Proof. Consider the continuous bilinear map

$$\ell: B \times Y \rightarrow Y, \quad (b, y) \mapsto \ell(b, y) = \Psi(b)y,$$

and its restriction to $A \times X$ which coincides with $(a, x) \mapsto \Phi(a)x \in X$ on $A \times X$ due to (22). We thus have the estimates

$$\|\ell(b, y)\|_Y \leq \|\Psi(b)\|_{\mathcal{L}(Y)} \|y\|_Y \leq \|\Psi\|_{\mathcal{L}(B, \mathcal{L}(Y))} \|b\|_B \|y\|_Y$$

and

$$\|\ell(a, x)\|_Y \leq \|\Phi(a)\|_{\mathcal{L}(X)} \|x\|_X \leq \|\Phi\|_{\mathcal{L}(A, \mathcal{L}(X))} \|a\|_A \|x\|_X.$$

Then, bilinear interpolation as in [54, Chapter 1.19.5] shows that ℓ gives rise to a continuous bilinear mapping $[A, B]_\theta \times [X, Y]_\theta \rightarrow [X, Y]_\theta$ whose norm is bounded by $\|\Phi\|_{\mathcal{L}(A, \mathcal{L}(X))}^{1-\theta} \|\Psi\|_{\mathcal{L}(B, \mathcal{L}(Y))}^\theta$. Hence, for $c \in [A, B]_\theta$ and $z \in [X, Y]_\theta$ we find

$$\|\Psi(c)z\|_{[X, Y]_\theta} = \|\ell(c, z)\|_{[X, Y]_\theta} \leq \|\Phi\|_{\mathcal{L}(A, \mathcal{L}(X))}^{1-\theta} \|\Psi\|_{\mathcal{L}(B, \mathcal{L}(Y))}^\theta \cdot \|c\|_{[A, B]_\theta} \|z\|_{[X, Y]_\theta}.$$

This shows that $\Psi \in \mathcal{L}([A, B]_\theta, \mathcal{L}([X, Y]_\theta))$, as desired. $\square$

A.2. Interpolation of Hölder spaces. The following result is well known for $E = \mathbb{R}^d$ or E being a domain with sufficiently smooth boundary, cf. e.g. [37, 54]. Since we did not find an explicit reference for a less regular setting in the literature we decided to sketch the short proof, although the appearing techniques are standard.

Proposition A.2. *For any nonempty compact set $E \subseteq \mathbb{R}^d$ there holds*

$$(C(E), C^{0,1}(E))_{\theta, \infty} = C^{0, \theta}(E), \quad \theta \in (0, 1),$$

and

$$C^{0, \vartheta}(E) \hookrightarrow [C(E), C^{0,1}(E)]_\theta \hookrightarrow C^{0, \theta}(E), \quad \theta \in (0, 1), \quad \vartheta \in (\theta, 1).$$

Proof. It suffices to prove the first statement. The second one follows from standard embeddings between real and complex interpolation scales. First, note that the claim holds for $E = \mathbb{R}^d$, see [37], below the proof of Example 1.1.8, for instance. To extend this to general E , we use the Whitney extension operator which provides a simultaneous extension operator $C(E) \rightarrow C(\mathbb{R}^d)$ and $C^{0,1}(E) \rightarrow C^{0,1}(\mathbb{R}^d)$, see [52, Chapter VI.2]. Then an application of the retraction-coretraction theorem [54, Theorem 1.2.4] finishes the proof. $\square$

A.3. Domains on interpolation spaces. The following two results are used in the bootstrapping argument in Section 5.

Lemma A.3. *Let A be a closed operator on a Banach space X with domain $\text{Dom}_X(A)$ such that $A: \text{Dom}_X(A) \rightarrow X$ is an isomorphism. Further, let $Y \hookrightarrow X$ be a subspace of X such that A is also an isomorphism $\text{Dom}_Y(A) \rightarrow Y$. Then for $\theta \in (0, 1)$ we have*

$$\text{Dom}_{[X, Y]_\theta}(A) = [\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta.$$

Proof. We equip $\text{Dom}_Z(A)$ with the norm $x \mapsto \|Ax\|_Z$ for any occurring space Z . Then A is an isometry between $\text{Dom}_X(A)$ and X and between $\text{Dom}_Y(A)$ and Y . Thus, by the functorial property of interpolation, A is nonexpansive, mapping $[\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta \rightarrow [X, Y]_\theta$, i.e.,

$$\|x\|_{\text{Dom}_{[X,Y]_\theta}(A)} = \|Ax\|_{[X,Y]_\theta} \leq \|x\|_{[\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta},$$

so

$$[\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta \hookrightarrow \text{Dom}_{[X,Y]_\theta}(A).$$

Quite analogously, A^{-1} is a contraction $[X, Y]_\theta \rightarrow [\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta$, which shows that

$$\text{Dom}_{[X,Y]_\theta}(A) \hookrightarrow [\text{Dom}_X(A), \text{Dom}_Y(A)]_\theta.$$

1 The claim follows; in fact, the stated equality is even an isometry with the chosen
2 norms. $\square$

3 **Lemma A.4.** *In addition to the assumptions of Lemma A.3, assume that A is*
4 *densely defined, positive, and admits bounded imaginary powers, each both on X*
5 *and Y , respectively. Then for $\theta, \rho \in (0, 1)$ there holds*

$$[[X, Y]_\theta, \text{Dom}_{[X,Y]_\theta}(A)]_\rho = [[X, \text{Dom}_X(A)]_\rho, [Y, \text{Dom}_Y(A)]_\rho]_\theta.$$

6 *Proof.* By interpolation, A has bounded imaginary powers on $[X, Y]_\theta$ as well. With
7 reiteration for fractional power domains as in [54, Theorem 1.15.3], we thus conclude

$$[[X, Y]_\theta, \text{Dom}_{[X,Y]_\theta}(A)]_\rho = \text{Dom}_{[X,Y]_\theta}(A^\rho).$$

8 Now, according to [54, Theorem 1.15.2 (e)], A^ρ is still an isomorphism $\text{Dom}_X(A^\rho) \rightarrow$
9 X and $\text{Dom}_Y(A^\rho) \rightarrow Y$, and hence fulfills the assumptions of the previous lemma.
10 Therefore, we obtain

$$\text{Dom}_{[X,Y]_\theta}(A^\rho) = [\text{Dom}_X(A^\rho), \text{Dom}_Y(A^\rho)]_\theta.$$

11 The claim follows from re-expanding the right-hand side via [54, Theorem 1.15.3].
12 $\square$

13 **Appendix B. Hölder-regularity for the elliptic operator in L^p .** In relation
14 to Definition 5.1 for the degree of Hölder-regularity for solutions to the elliptic
15 problem associated to $-\nabla \cdot \mu \nabla + 1$ in L^p , we provide some examples for explicit κ
16 in some rather nonsmooth cases:

Example B.1. Let Ω be a convex domain, let μ be Lipschitz-continuous, and let $\Gamma_D = \partial\Omega$ or $\Gamma_D = \emptyset$. From [21, Theorems 3.2.1.2&3.2.1.3] it is well known that

$$-\nabla \cdot \mu \nabla + 1: H^2 \cap \ker(B) \rightarrow L^2,$$

is a topological isomorphism, with $B = \text{tr}$ if $\Gamma_D = \partial\Omega$, or $B = \nu \cdot \mu \nabla$, the co-normal derivative on $\partial\Omega$ associated with μ , if $\Gamma_D = \emptyset$. Hereby, we consider $\ker(B)$ as a subspaces of $W^{1,1}$ and $W^{2,1}$, respectively. Every convex domain is locally uniform [53, Proposition 3.8]. Consequently, there is a degree-independent Sobolev extension operator for Ω [49], and the families $(L^q)_{q \in (1, \infty)}$ and $(W^{2,q} \cap \ker(B))_{q \in (1, \infty)}$ form interpolation scales such that

$$-\nabla \cdot \mu \nabla + 1: W^{2,q} \cap \ker(B) \rightarrow L^q \tag{23}$$

17 is a bounded linear operator for every $q \in (1, \infty)$. It follows from Sneiberg's ex-
18 trapolation theorem, see [42, Theorem 3.1], that there is some $\bar{q} > 2$ such that (23)

1 is still an isomorphism for $q \in [2, \bar{q}]$. Sobolev embeddings therefore imply that we
 2 can choose $\kappa > 1$ in dimension $d = 2$ and $\kappa > \frac{1}{2}$ in dimension $d = 3$.

3 The following example is concerned with polygonal domains in $\mathbb{R}^2$, constant
 4 coefficients, and possibly mixed boundary conditions:

5 **Example B.2.** Let $\Omega \subset \mathbb{R}^2$ be a polygon with vertices S_j , $j = 1, \dots, N$. By ω_j we
 6 denote the interior angle of Ω at S_j . The type of boundary condition on each edge
 7 has to be fixed (homogeneous Dirichlet or Neumann), which is not a restriction,
 8 because “artificial” vertices with $\omega_j = \pi$ are allowed. In order to keep the notation
 9 as simple as possible, we restrict ourselves to the Laplacian, i.e., $\mu = \text{id}$, for this
 10 example.

Given $g \in L^p$, let $f \in H_D^1$ denote the solution of $-\Delta f + f = g$. From [21, Theorem 4.4.3.7] we infer that there is a regular part of the solution $f_0 \in W^{2,p} \cap W_D^{1,p}$ such that the remainder $f - f_0$ can be expressed in polar coordinates (r_j, θ_j) centered at S_j as follows:

$$(f - f_0)(r_j, \theta_j) = \sum_{1 \leq j \leq N} \eta_j(r_j, \theta_j) \sum_{0 < \lambda_{j,m} < 2/p'} c_{j,m} r_j^{\lambda_{j,m}} s_{j,m}(\theta_j) \quad (24)$$

Herein, η_j denote smooth cut-off functions with their support near S_j , the $c_{j,m}$ are scalar coefficients, and the $s_{j,m}$ are smooth. The exponents $\lambda_{j,m}$ correspond to the eigenvalues of certain Sturm-Liouville eigenvalue problems, and can be determined analytically, cf. [21, p.220]:

$$\begin{aligned} \lambda_{j,m} &= \frac{\pi m}{\omega_j} && \text{for pure Dirichlet or Neumann boundary conditions around } S_j, \\ \lambda_{j,m} &= \frac{\pi m}{2\omega_j} && \text{if boundary conditions change at } S_j. \end{aligned}$$

11 Therefore, any $f \in \text{Dom}_{L^p}(-\Delta + 1)$ consists of a $W^{2,p}$ -part f_0 , which is then in
 12 $C^{1,1-d/p}(\bar{\Omega})$, and a singular part given by (24). From [21, Theorem 6.2.10] we infer
 13 that this singular part is Hölder-continuous with its degree given by the minimal
 14 $\lambda_{j,m}$ appearing in the sum. Hence, we conclude

$$\kappa = \min \left\{ 2 - \frac{d}{p}, \min_{\substack{\omega_j > \pi \\ \text{unif. BC's at } j}} \frac{\pi}{\omega_j}, \min_{\substack{\omega_j > \frac{\pi}{2} \\ \text{mixed BC's at } j}} \frac{\pi}{2\omega_j} \right\}$$

15 in this example. Note that using [21, Theorem 5.2.7] also non-constant, but Lipschitz-
 16 continuous μ can be treated in case of pure homogeneous Dirichlet boundary con-
 17 ditions.

18 Following [45], Example B.2 can be extended to a *scalar* coefficient function μ
 19 that is constant with respect to a polygonal partition of Ω :

Example B.3. Let $\Omega \subset \mathbb{R}^2$ be a polygon and μ be a scalar function that is piecewise constant with respect to a partition of Ω into subdomains given by polygons. In [44, Theorem 2.27] a decomposition of $f \in \text{Dom}_{L^p}(-\nabla \cdot \mu \nabla + 1)$ into a $W^{2,p}$ -part and a singular part, similar to the previous example, is obtained. If μ is non-constant, then singularities can occur at any vertex of the boundary and also along the discontinuities of μ inside the domain. Again, the “bad” contributions to f arise from terms of type $r^\lambda s(\theta)$ (in polar coordinates) with λ to be determined from certain Sturm-Liouville eigenvalue problems, cf. [45]. In order to have lower bounds on these eigenvalues depending only on the underlying geometry, but not on the

actual values of μ , we require the following restriction: For any $x \in \bar{\Omega}$ the number of different values of μ (“materials”) adjacent to x plus the number of boundary conditions at x does not exceed 3. Under this assumption the following eigenvalue estimates are provided in [45], see also [14, Section 8]:

$$\begin{aligned} \lambda_x &> \frac{\pi}{2\omega_x} & x \in \partial\Omega, \text{ two materials, uniform boundary condition,} \\ \lambda_x &> \frac{\pi}{2\omega_x} & x \in \partial\Omega, \text{ one material, boundary condition changing,} \\ \lambda_x &> \frac{1}{2} & x \in \Omega, \text{ two adjacent materials,} \\ \lambda_x &> \frac{1}{4} & x \in \Omega, \text{ three adjacent materials.} \end{aligned}$$

1 Hereby, ω_x denotes the interior angle of Ω at the (possibly artificial) vertex $x \in \partial\Omega$.
 2 The remaining case, i.e., a point $x \in \partial\Omega$ with only one adjacent material and
 3 uniform boundary condition, has already been dealt with in Example B.2. For the
 4 details and similar results under less strict assumptions we refer to [45]. As in the
 5 previous example, the maximal value for κ is determined by the minimal λ_x for
 6 $x \in \bar{\Omega}$.

7 In three space dimensions the situation is more involved. For example, in elliptic
 8 problems on polyhedral domains there are singularities arising from both vertices
 9 and edges. We refer to [15, 16, 22] for an overview. For a constant coefficient
 10 function μ and pure homogeneous Dirichlet boundary conditions, a decomposition
 11 into a regular and a singular part similar to the one in Example B.2 is still possible,
 12 see e.g. the exposition in the proof of [55, Lemma 2.3]. The resulting degree
 13 of Hölder-continuity, however, depends on the eigenvalues of the Laplace-Beltrami
 14 operator on certain spherical polygons and seems to be difficult to determine. We
 15 thus do not go into further details.

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18 References.

- 19 [1] H. Amann. “Maximal regularity for nonautonomous evolution equations”. In:
 20 *Adv. Nonlinear Stud.* 4.4 (2004), pp. 417–430. DOI: [10.1515/ans-2004-0404](https://doi.org/10.1515/ans-2004-0404).
- 21 [2] H. Amann. “Nonautonomous parabolic equations involving measures”. In:
 22 *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 306. Kraev.
 23 Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 34 (2003), pp. 16–52, 229.
 24 DOI: [10.1007/s10958-005-0376-8](https://doi.org/10.1007/s10958-005-0376-8).
- 25 [3] H. Amann. *Linear and quasilinear parabolic problems. Vol. I.* Vol. 89. Mono-
 26 graphs in Mathematics. Abstract linear theory. Birkhäuser Boston, Inc., Boston,
 27 MA, 1995, pp. xxxvi+335. DOI: [10.1007/978-3-0348-9221-6](https://doi.org/10.1007/978-3-0348-9221-6).
- 28 [4] H. Amann. “Quasilinear parabolic problems via maximal regularity”. In: *Adv.*
 29 *Differential Equations* 10.10 (2005), pp. 1081–1110.
- 30 [5] P. Auscher et al. “The square root problem for second-order, divergence form
 31 operators with mixed boundary conditions on L^p ”. In: *Journal of Evolution*
 32 *Equations* 15.1 (Oct. 2014), pp. 165–208. DOI: [10.1007/s00028-014-0255-1](https://doi.org/10.1007/s00028-014-0255-1).
- 33 [6] S. Bechtel and M. Egert. “Interpolation theory for Sobolev functions with
 34 partially vanishing trace on irregular open sets”. In: *J. Fourier Anal. Appl.*
 35 25.5 (2019), pp. 2733–2781. DOI: [10.1007/s00041-019-09681-1](https://doi.org/10.1007/s00041-019-09681-1).

- 1 [7] S. Bechtel, M. Egert, and R. Haller-Dintelmann. “The Kato square root prob-
2 lem on locally uniform domains”. In: *Advances in Mathematics* 375 (Dec.
3 2020), p. 107410. DOI: [10.1016/j.aim.2020.107410](https://doi.org/10.1016/j.aim.2020.107410).
- 4 [8] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Grundlehren
5 der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New
6 York, 1976, pp. x+207.
- 7 [9] L. Bonifacius and I. Neitzel. “Second order optimality conditions for optimal
8 control of quasilinear parabolic equations”. In: *Math. Control Relat. Fields*
9 8.1 (2018), pp. 1–34. DOI: [10.3934/mcrf.2018001](https://doi.org/10.3934/mcrf.2018001).
- 10 [10] K. Brewster et al. “Extending Sobolev functions with partially vanishing
11 traces from locally (ε, δ) -domains and applications to mixed boundary prob-
12 lems”. In: *Journal of Functional Analysis* 266.7 (Apr. 2014), pp. 4314–4421.
13 DOI: [10.1016/j.jfa.2014.02.001](https://doi.org/10.1016/j.jfa.2014.02.001).
- 14 [11] E. Casas. “Boundary control of semilinear elliptic equations with pointwise
15 state constraints”. In: *SIAM J. Control Optim.* 31.4 (1993), pp. 993–1006.
16 DOI: [10.1137/0331044](https://doi.org/10.1137/0331044).
- 17 [12] E. Casas and K. Chrysafinos. “Analysis and optimal control of some quasilin-
18 ear parabolic equations”. In: *Mathematical Control and Related Fields* 8.3-4
19 (2018), pp. 607–623. DOI: [10.3934/mcrf.2018025](https://doi.org/10.3934/mcrf.2018025).
- 20 [13] E. Casas, J. C. de los Reyes, and F. Tröltzsch. “Sufficient second-order opti-
21 mality conditions for semilinear control problems with pointwise state con-
22 straints”. In: *SIAM J. Optim.* 19.2 (2008), pp. 616–643. DOI: [10.1137 /
23 07068240X](https://doi.org/10.1137/07068240X).
- 24 [14] M. Costabel, M. Dauge, and S. Nicaise. “Singularities of Maxwell interface
25 problems”. In: *M2AN Math. Model. Numer. Anal.* 33.3 (1999), pp. 627–649.
26 DOI: [10.1051/m2an:1999155](https://doi.org/10.1051/m2an:1999155).
- 27 [15] M. Dauge. “Neumann and mixed problems on curvilinear polyhedra”. In:
28 *Integral Equations Operator Theory* 15.2 (1992), pp. 227–261. DOI: [10.1007/
29 BF01204238](https://doi.org/10.1007/BF01204238).
- 30 [16] M. Dauge. “Singularities of corner problems and problems of corner singulari-
31 ties”. In: *Actes du 30ème Congrès d’Analyse Numérique: CANum ’98 (Arles,
32 1998)*. Vol. 6. ESAIM Proc. Soc. Math. Appl. Indust., Paris, 1999, pp. 19–40.
33 DOI: [10.1051/proc:1999044](https://doi.org/10.1051/proc:1999044).
- 34 [17] K. Disser and J. Rehberg. “The 3D transient semiconductor equations with
35 gradient-dependent and interfacial recombination”. In: *Mathematical Models
36 and Methods in Applied Sciences* 29.10 (Sept. 2019), pp. 1819–1851. DOI:
37 [10.1142/s0218202519500350](https://doi.org/10.1142/s0218202519500350).
- 38 [18] K. Disser, A. F. M. ter Elst, and J. Rehberg. “On maximal parabolic regularity
39 for non-autonomous parabolic operators”. In: *J. Differential Equations* 262.3
40 (2017), pp. 2039–2072. DOI: [10.1016/j.jde.2016.10.033](https://doi.org/10.1016/j.jde.2016.10.033).
- 41 [19] K. Disser, H.-C. Kaiser, and J. Rehberg. “Optimal Sobolev regularity for
42 linear second-order divergence elliptic operators occurring in real-world prob-
43 lems”. In: *SIAM J. Math. Anal.* 47.3 (2015), pp. 1719–1746. DOI: [10.1137/
44 140982969](https://doi.org/10.1137/140982969).
- 45 [20] J. Elschner, J. Rehberg, and G. Schmidt. “Optimal regularity for elliptic
46 transmission problems including C^1 interfaces”. In: *Interfaces Free Bound.*
47 9.2 (2007), pp. 233–252. DOI: [10.4171/IFB/163](https://doi.org/10.4171/IFB/163).

- 1 [21] P. Grisvard. *Elliptic problems in nonsmooth domains*. Vol. 24. Monographs
2 and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston,
3 MA, 1985, pp. xiv+410.
- 4 [22] P. Grisvard. “Singular behavior of elliptic problems in non-Hilbertian Sobolev
5 spaces”. In: *J. Math. Pures Appl. (9)* 74.1 (1995), pp. 3–33.
- 6 [23] K. Gröger. “A $W^{1,p}$ -estimate for solutions to mixed boundary value problems
7 for second order elliptic differential equations”. In: *Math. Ann.* 283.4 (1989),
8 pp. 679–687. DOI: [10.1007/BF01442860](https://doi.org/10.1007/BF01442860).
- 9 [24] R. Haller-Dintelmann et al. “Hölder continuity and optimal control for non-
10 smooth elliptic problems”. In: *Appl. Math. Optim.* 60.3 (2009), pp. 397–428.
11 DOI: [10.1007/s00245-009-9077-x](https://doi.org/10.1007/s00245-009-9077-x).
- 12 [25] R. Haller-Dintelmann and J. Rehberg. “Maximal parabolic regularity for di-
13 vergence operators including mixed boundary conditions”. In: *J. Differential*
14 *Equations* 247.5 (2009), pp. 1354–1396. DOI: [10.1016/j.jde.2009.06.001](https://doi.org/10.1016/j.jde.2009.06.001).
- 15 [26] R. Haller-Dintelmann et al. “Elliptic and parabolic regularity for second-order
16 divergence operators with mixed boundary conditions”. In: *Math. Methods*
17 *Appl. Sci.* 39.17 (2016), pp. 5007–5026. DOI: [10.1002/mma.3484](https://doi.org/10.1002/mma.3484).
- 18 [27] J. Heinonen. *Lectures on Lipschitz analysis*. Vol. 100. Report. University of
19 Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä,
20 Jyväskylä, 2005, pp. ii+77.
- 21 [28] D. Henry. *Geometric theory of semilinear parabolic equations*. Vol. 840. Lec-
22 ture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1981, pp. iv+348.
- 23 [29] M. Hieber and J. Prüss. “Thermodynamical Consistent Modeling and Analysis
24 of Nematic Liquid Crystal Flows”. In: *Mathematical Fluid Dynamics, Present*
25 *and Future*. Springer Japan, 2016, pp. 433–459. DOI: [10.1007/978-4-431-](https://doi.org/10.1007/978-4-431-56457-7_15)
26 [56457-7_15](https://doi.org/10.1007/978-4-431-56457-7_15).
- 27 [30] M. Hieber and J. Rehberg. “Quasilinear Parabolic Systems with Mixed Bound-
28 ary Conditions on Nonsmooth Domains”. In: *SIAM Journal on Mathematical*
29 *Analysis* 40.1 (Jan. 2008), pp. 292–305. DOI: [10.1137/070683829](https://doi.org/10.1137/070683829).
- 30 [31] M. Hieber et al. “Dynamics of nematic liquid crystal flows: The quasilinear
31 approach”. In: *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*
32 33.2 (Apr. 2016), pp. 397–408. DOI: [10.1016/j.anihpc.2014.11.001](https://doi.org/10.1016/j.anihpc.2014.11.001).
- 33 [32] F. Hoppe and I. Neitzel. “Optimal Control of Quasilinear Parabolic PDEs
34 with State-Constraints”. In: *SIAM Journal on Control and Optimization* 60.1
35 (Jan. 2022), pp. 330–354. DOI: [10.1137/20m1383951](https://doi.org/10.1137/20m1383951).
- 36 [33] A. Jonsson and H. Wallin. *Function spaces on subsets of $\mathbb{R}^n$* . Harwood Aca-
37 demic Publishers, Chur-London-Paris-Utrecht-New York, 1984.
- 38 [34] M. Köhne, J. Prüss, and M. Wilke. “On quasilinear parabolic evolution equa-
39 tions in weighted L^p -spaces”. In: *Journal of Evolution Equations* 10.2 (Feb.
40 2010), pp. 443–463. DOI: [10.1007/s00028-010-0056-0](https://doi.org/10.1007/s00028-010-0056-0).
- 41 [35] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’čeva. *Linear and quasi-*
42 *linear equations of parabolic type*. Translated from the Russian by S. Smith.
43 Translations of Mathematical Monographs, Vol. 23. American Mathematical
44 Society, Providence, R.I., 1968, pp. xi+648.
- 45 [36] J. LeCrone, J. Prüss, and M. Wilke. “On quasilinear parabolic evolution equa-
46 tions in weighted L^p -spaces II”. In: *Journal of Evolution Equations* 14.3 (Mar.
47 2014), pp. 509–533. DOI: [10.1007/s00028-014-0226-6](https://doi.org/10.1007/s00028-014-0226-6).

- 1 [37] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*.
 2 Modern Birkhäuser Classics. [2013 reprint of the 1995 original] [MR1329547].
 3 Birkhäuser/Springer Basel AG, Basel, 1995, pp. xviii+424.
- 4 [38] A. Lunardi. *Interpolation theory*. Second. Appunti. Scuola Normale Superiore
 5 di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New
 6 Series)]. Edizioni della Normale, Pisa, 2009, pp. xiv+191.
- 7 [39] V. Maz'ya. *Sobolev Spaces*. Springer Berlin Heidelberg, 2011. DOI: [10.1007/
 8 978-3-642-15564-2](https://doi.org/10.1007/978-3-642-15564-2).
- 9 [40] H. Meinlschmidt, C. Meyer, and J. Rehberg. “Optimal control of the ther-
 10 mistor problem in three spatial dimensions, Part 1: Existence of optimal
 11 solutions”. In: *SIAM J. Control Optim.* 55.5 (2017), pp. 2876–2904. DOI:
 12 [10.1137/16M1072644](https://doi.org/10.1137/16M1072644).
- 13 [41] H. Meinlschmidt, C. Meyer, and J. Rehberg. “Optimal control of the ther-
 14 mistor problem in three spatial dimensions, Part 2: Optimality conditions”.
 15 In: *SIAM J. Control Optim.* 55.4 (2017), pp. 2368–2392. DOI: [10.1137/
 16 16M1072656](https://doi.org/10.1137/16M1072656).
- 17 [42] H. Meinlschmidt and J. Rehberg. “Extrapolated elliptic regularity and ap-
 18 plication to the van Roosbroeck system of semiconductor equations”. In: *J.
 19 Differential Equations* 280 (2021), pp. 375–404. DOI: [10.1016/j.jde.2021.
 20 01.032](https://doi.org/10.1016/j.jde.2021.01.032).
- 21 [43] H. Meinlschmidt and J. Rehberg. “Hölder-estimates for non-autonomous par-
 22 abolic problems with rough data”. In: *Evol. Equ. Control Theory* 5.1 (2016),
 23 pp. 147–184. DOI: [10.3934/eect.2016.5.147](https://doi.org/10.3934/eect.2016.5.147).
- 24 [44] S. Nicaise. *Polygonal interface problems*. Vol. 39. Methoden und Verfahren der
 25 Mathematischen Physik [Methods and Procedures in Mathematical Physics].
 26 Verlag Peter D. Lang, Frankfurt am Main, 1993, pp. ii+250.
- 27 [45] M. Petzoldt. “Regularity results for Laplace interface problems in two dimen-
 28 sions”. English. In: *Z. Anal. Anwend.* 20.2 (2001), pp. 431–455.
- 29 [46] P. Portal and Ž. Štrkalj. “Pseudodifferential operators on Bochner spaces
 30 and an application”. In: *Math. Z.* 253.4 (2006), pp. 805–819. DOI: [10.1007/
 31 s00209-006-0934-x](https://doi.org/10.1007/s00209-006-0934-x).
- 32 [47] J. Prüss. “Maximal regularity for evolution equations in L_p -spaces”. In: *Conf.
 33 Semin. Mat. Univ. Bari* 285 (2002), 1–39 (2003).
- 34 [48] J. Prüss and R. Schnaubelt. “Solvability and maximal regularity of parabolic
 35 evolution equations with coefficients continuous in time”. In: *J. Math. Anal.
 36 Appl.* 256.2 (2001), pp. 405–430. DOI: [10.1006/jmaa.2000.7247](https://doi.org/10.1006/jmaa.2000.7247).
- 37 [49] L. G. Rogers. “Degree-independent Sobolev extension on locally uniform do-
 38 mains”. In: *J. Funct. Anal.* 235.2 (2006), pp. 619–665. DOI: [10.1016/j.jfa.
 39 2005.11.013](https://doi.org/10.1016/j.jfa.2005.11.013).
- 40 [50] S. Selberherr. *Analysis and Simulation of Semiconductor Devices*. Springer-
 41 Verlag, 1984.
- 42 [51] E. Shamir. “Regularization of mixed second-order elliptic problems”. In: *Is-
 43 rael Journal of Mathematics* 6.2 (Apr. 1968), pp. 150–168. DOI: [10.1007/
 44 bf02760180](https://doi.org/10.1007/bf02760180).
- 45 [52] E. M. Stein. *Singular integrals and differentiability properties of functions*.
 46 Princeton Mathematical Series, No. 30. Princeton University Press, Princeton,
 47 N.J., 1970, pp. xiv+290.

- 1 [53] H. Triebel. *Function spaces and wavelets on domains*. Vol. 7. EMS Tracts in
2 Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. x+256.
3 DOI: [10.4171/019](https://doi.org/10.4171/019).
- 4 [54] H. Triebel. *Interpolation theory, function spaces, differential operators*. Sec-
5 ond. Johann Ambrosius Barth, Heidelberg, 1995, p. 532.
- 6 [55] W. Wollner. “Optimal control of elliptic equations with pointwise constraints
7 on the gradient of the state in nonsmooth polygonal domains”. In: *SIAM J.*
8 *Control Optim.* 50.4 (2012), pp. 2117–2129. DOI: [10.1137/110836419](https://doi.org/10.1137/110836419).
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