
Exercise 1 (Existence of globally optimal solutions)**.** Determine whether the following optimization problems in function spaces admit a globally optimal solution.

$$\min_{u \in C([0,1])} \int_0^1 u(x)^2 \, \mathrm{d}x \quad \text{s.t.} \quad u(1) = 1, \tag{P1}$$

where $C([0,1])$ is the Banach space of all continuous functions $u\colon [0,1] \to \mathbb{R}$ equipped with the norm $\|u\|_\infty := \max_{x \in [0,1]} |u(x)|$,

$$\min_{u \in L^2(0,1)} \; -\int_0^1 x\, u(x)^2 \, \mathrm{d}x \quad \text{s.t.} \quad \|u\|_{L^2(0,1)} \le 1, \tag{P2}$$

and

$$\max_{y \in H^1(0,1)} \|y\|_{L^\infty(0,1)} \quad \text{s.t.} \quad \|y\|_{H^1(0,1)} \le 2, \tag{P3}$$

where $H^1(0,1)$ is the Sobolev (Hilbert) space $H^1(0,1) := \{y \in L^2(0,1)\colon y' \in L^2(0,1)\}$ equipped with the norm $\|y\|_{H^1(0,1)} := \|y\|_{L^2(0,1)} + \|y'\|_{L^2(0,1)}$.

Hint: The natural embedding $W^{1,2}(0,1) = H^1(0,1) \hookrightarrow C([0,1])$ induced by the identity mapping $u \mapsto u$ is a *compact* linear operator, see Exercise 4 below.

Exercise 2 (Continuity of superposition operators in Lebesgue-spaces)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function and let X be a function space consisting of real-valued functions defined on a bounded open set $\Omega \subseteq \mathbb{R}^n$. Then the *superposition* or *Nemytskii* operator F (on X) induced by f is given by the mapping $X \ni u \mapsto f \circ u$, i.e., $F(u)(x) := f(u(x))$ as a function of $x \in \Omega$.

(a) Let $1 \le p, q < \infty$ and assume that f is continuous and satisfies

$$\big|f(t)\big| \le C\big(|t|^{\frac{p}{q}} + 1\big) \tag{1}$$

for some constant $C \ge 0$. Show that F is a sequentially continuous mapping from $L^p(\Omega)$ to $L^q(\Omega)$.

Hint: From the proof of the Riesz-Fischer theorem (completeness of L^p): Every L^p convergent sequence admits a subsequence which converges in a pointwise sense almost everywhere and which is uniformly bounded by an L^p function.

(b) Let $\Omega = (0, 1)$ and assume that F is weakly sequentially continuous from $L^p(\Omega)$ to $L^q(\Omega)$, i.e., if $u_k \rightharpoonup u$ in $L^p(\Omega)$, then $F(u_k) \rightharpoonup F(u)$ in $L^q(\Omega)$. Show that f must already be an *affine-linear* function.

Hint: Use Rademacher's functions from Exercise 3.

(c) Let $1 < p < \infty$, let Ω be bounded with a Lipschitz boundary, and assume that f is Lipschitz-continuous (in particular, f satisfies (1) for $q = p$). Show that F is weakly sequentially continuous as a mapping from $W^{1,p}(\Omega)$ to itself. Discuss the difference to the previous case.

Hint: The properties of Ω imply the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ (this is the Rellich-Kondrachov theorem).

Exercise 3 (An interesting family of functions (Rademacher))**.** Let $1 < p < \infty$ and let $f \in L^p_{\text{loc}}(\mathbb{R})$, that is, $f \in L^p(K)$ for every compact set $K \Subset \mathbb{R}$. Assume that $f(x + T) = f(x)$ for almost every $x \in \mathbb{R}$, so f is T-periodic with $T > 0$. Set

$$\overline{f} := T^{-1} \int_0^T f(x)\, \mathrm{d}x$$

and consider the sequence $(u_k) \subset L^p(0, 1)$ defined by

$$u_k(x) := f(kx), \quad x \in (0, 1).$$

(a) Show that $u_k \rightharpoonup \overline{f}$ in $L^p(0, 1)$.

Hint: It is sufficient to show the assertion for dual pairs with step functions in $L^{p'}(0, 1)$ (why?).

(b) Examine the following examples:

(i) $f(x) = \sin(x)$,

(ii) f is 1-periodic given by

$$f(x) := \begin{cases} \alpha & \text{if } x \in (0, \frac{1}{2}), \\ \beta & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

for $\alpha, \beta \in \mathbb{R}$. Such functions are called *Rademacher's functions*.

Exercise 4 (A particularly important compact embedding (Sobolev))**.** In Exercise 2, we have already used compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. In fact, the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for such domains whenever $\frac{1}{n} + \frac{1}{q} > \frac{1}{p}$. For $p > n$, even more is true, which we establish exemplarily for $n = 1$ and $\Omega = (0, 1)$.

(a) Show that, for every $1 \leq p \leq \infty$, the space $W^{1,p}(0,1)$ is a subset of $L^\infty(0,1)$ and that the embedding $W^{1,p}(0,1) \hookrightarrow L^\infty(0,1)$ is continuous, so

$$\|u\|_{L^\infty(0,1)} \leq C\|u\|_{W^{1,p}(0,1)} = C\big(\|u\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)}\big)$$

for some constant $C > 0$ independent of u.

Hint: The smooth functions $C^\infty([0,1])$ on $[0,1]$ are dense in $W^{1,p}(0,1)$.

(b) Refine the previous embedding by proving that for $p > 1$ we even have $W^{1,p}(0,1) \hookrightarrow C^{0,1-\frac{1}{p}}([0,1])$, where

$$C^{0,\alpha}([0,1]) := \Big\{u \in C([0,1])\colon \|u\|_{C^{0,\alpha}([0,1])} < \infty\Big\}$$

with

$$\|u\|_{C^{0,\alpha}([0,1])} := \|u\|_{C([0,1])} + \sup_{x \neq y \in [0,1]} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is the α-*Hölder* space for $0 < \alpha \leq 1$.

(c) Prove that every bounded sequence in $C^{0,\alpha}([0,1])$ admits a uniformly convergent subsequence, or equivalently, that the embedding $C^{0,\alpha}([0,1]) \hookrightarrow C([0,1])$ is compact.

Hint: Recall the Arzelà-Ascoli theorem.

(d) Infer that for $p > 1$, the space $W^{1,p}(0,1)$ embeds *compactly* into every Hölder space $C^{0,\alpha}([0,1])$ for $0 < \alpha < 1 - \frac{1}{p}$ and into the space of uniformly continuous functions $C([0,1])$.