Optimization in Function Spaces Exercise sheet **2**

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Exercise 1 (Differentiability in Banach spaces)**.** Verify the assertions from Example 3.3 in the lecture notes and some more:

(a) Show that every bounded linear operator $A \in \mathcal{L}(X;Y)$ is F-differentiable and its derivative in every point $x \in X$ is given by A itself.

(b) Let $a\colon X \times X \to \mathbb{R}$ be a symmetric continuous bilinear form. Prove that the quadratic form given by $X \ni u \mapsto \frac{1}{2}a(u,u) \in \mathbb{R}$ is F-differentiable and its derivative in $u \in X$ is given by $h \mapsto a(u,h)$. Apply this to the function $u \mapsto \frac{1}{2}\|u\|_X^2$ for a Hilbert space X.

Hint: The bilinear form a is continuous if and only if there exists a number $C \geq 0$ such that $|a(u,v)| \leq C\|u\|_X\|v\|_X$ for all $u, v \in X$.

(c) We consider the superposition operator Ψ induced by $\sin\colon \mathbb{R} \to \mathbb{R}$.

(i) Show that Ψ is F-differentiable as a mapping from $L^\infty(0,1)$ into itself with the derivative given by $h \mapsto \cos(y)h$ for every $y \in L^\infty(0,1)$ (can you generalize this assertion to other inducing functions?), ...

Hint: Calculate pointwisely and use exact first-order Taylor approximation in integrated form.

(ii) ... but Ψ is *not* F-differentiable as a mapping from $L^p(0,1)$ into itself for any $1 \leq p < \infty$.

Hint: Determine the residual of the first-order approximation exactly for suitably chosen step functions h. Choosing $y \equiv 0$ also helps to clear the fog.

(iii) Guess and prove the relation between p and q such that Ψ is F-differentiable as a mapping from $L^q(0,1)$ to $L^p(0,1)$.

Hint: You may use without proof that a superposition operator mapping $L^q(0,1)$ to $L^p(0,1)$ for $1 \leq p, q < \infty$ is always automatically continuous.

(d) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Prove that the superposition operator Ξ induced by the real function $\varphi(t) := t^3$ is F-differentiable when considered as a mapping from $L^6(\Omega)$ to $L^2(\Omega)$ and its derivative is given by $L^6(\Omega) \ni h \mapsto 3y^2h \in L^2(\Omega)$.

Exercise 2 (Closedness of the tangential cone)**.** Let X be a Banach space and let $x \in M \subseteq X$. Show that the contingent cone $T(M,x)$ is closed.

Exercise 3 (Linearizing cone)**.** Verify Remark 3.11 in the lecture notes, that is: The linearizing cone for the NLP

$$\min f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0,$$

with $f\colon \mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $h\colon \mathbb{R}^n \to \mathbb{R}^p$, is given by

$$T_\ell(G, K, x) = \Big\{ d \in \mathbb{R}^n \colon \nabla h(x)^T d = 0,\ \nabla g_i(x)^T d \le 0 \text{ for } i \in \mathcal{A}(x) \Big\}$$

for a feasible point x.

Exercise 4 (Optimal control problem in reduced form)**.** Consider the basic optimal control problem

$$\min_{(y,u) \in Y \times U} J(y, u) \quad \text{s.t.} \quad E(y, u) = 0$$

with Y, U and Z Banach spaces and $J\colon Y \times U \to \mathbb{R}$ and $E\colon Y \times U \to Z$ (continuously) F-differentiable. Assume that for every $u \in U$ there exists a unique $y = y(u) \in Y$ such that $E(y(u), u) = 0$. Then we can reduce the above optimal control problem to the unrestricted optimization problem

$$\min_{u \in U} j(u) := J(y(u), u). \tag{ROCP}$$

In this exercise, we investigate the *control-to-state* operator $U \ni u \mapsto y(u) \in Y$ more in depth. We additionally assume that $E'_y(y(u), u) = (\partial_y E)(y(u), u) \in \mathcal{L}(Y; Z)$ is continuously invertible for every $u \in U$, so the inverse operator exists and is also linear and continuous.

(a) Use the *implicit function theorem* to show that y is (continuously) F-differentiable and determine an explicit formula for $y'(u)$ from $E(y(u), u) = 0$ for all $u \in U$.

(b) Give an expression for the *sensitivity* $j'(u)h$ in direction $h \in U$ using the derived formula for $y'(u)$.

(c) Show that we can represent the total derivative $j'(u)$ by

$$j'(u) = y'(u)^* J'_y(y(u), u) + J'_u(y(u), u) = E'_u(y(u), u)^* p + J'_u(y(u), u),$$

where $p = p(u) \in Z^*$ is the *adjoint state* satisfying the *adjoint equation*

$$E'_y(y(u), u)^* p = -J'_y(y(u), u).$$

(d) Let lastly Y, U and Z be finite-dimensional. We imagine this to originate from a discretization of the infinite-dimensional problem, so the underlying space dimensions n_Y, n_u and n_Z may be very high and taking inverse matrices is not an option. Compare the effort needed to compute the total derivative $j'(u)$ (or $\nabla j(u)$ for that matter) using the sensitivity approach and the adjoint approach, respectively.