
Exercise 1 (Linear operators in multiple components, Jacobian, KKT-conditions)**.** Let $X_1, \dots, X_n$ and $Z_1, \dots, Z_m$ be Banach spaces and set $X := X_1 \times \cdots \times X_n$ as well as $Z := Z_1 \times \cdots \times Z_m$. Consider an operator $A \in \mathcal{L}(X; Z)$.

(a) Show that A uniquely corresponds to an $m \times n$-operator-matrix $\mathcal{A} = (A_{ij})$ of continuous linear operators $A_{ij} \in \mathcal{L}(X_j; Z_i)$ such that

$$Ax = \mathcal{A}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{for } x = (x_1, \dots, x_n) \quad \text{with } x_i \in X_i,$$

and that $A \mapsto \sum_{i=1}^m \max_{1 \le j \le n} \|A_{ij}\|_{\mathcal{L}(X_j;Z_i)}$ is an equivalent norm to $\|\cdot\|_{\mathcal{L}(X;Z)}$.

(b) Show that $X^* = X_1^* \times \cdots \times X_n^*$ and $Z^* = Z_1^* \times \cdots \times Z_m^*$ and determine the operator-matrix corresponding to $A^* \in \mathcal{L}(Z^*; X^*)$.

(c) Let $G \colon X \to Z$ be F-differentiable around $\bar{x} \in X$. Show that the operator-matrix $\mathcal{G}'(\bar{x})$ of $G'(\bar{x})$ is exactly a generalized Jacobian matrix of G in $\bar{x}$.

(d) Let $(\bar{y}, \bar{u})$ be a regular point of the control-constrained optimal control problem

$$\min_{(y,u) \in Y \times U} J(y,u) \quad \text{s.t.} \quad E(y,u) = 0, \quad u \in U_{\text{ad}},$$

where $J \colon Y \times U \to \mathbb{R}$ and $E \colon Y \times U \to Z$ are F-differentiable, Y, U, Z are Banach spaces, and U_{ad} is closed and convex. Apply the above results to the multiplier rule in the KKT-conditions of this problem for $(\bar{y}, \bar{u})$.

Remark: Recall (or verify) that every norm $\|\cdot\|_\alpha$ on $\mathbb{R}^n$ constructed in the form $\|\mathbf{x}\| = f(|\mathbf{x}_1|, \dots, |\mathbf{x}_n|)$ for $\mathbf{x} \in \mathbb{R}^n$ also gives rise to a norm $\|x\|_{\alpha,X} = f(\|x_1\|_{X_1}, \dots, \|x_n\|_{X_n})$ on X, for example $\|(x_1, \dots, x_n)\|_{1,X} := \sum_{i=1}^n \|x_i\|_{X_i}$, and all these norms are equivalent because the ones on $\mathbb{R}^n$ are; an analogous result of course holds for Z and $\mathbb{R}^m$. For convenience, we always choose the norm induced by the $\|\cdot\|_1$-norm on the finite-dimensional space.

Exercise 2 (Lax-Milgram lemma and divergence-gradient operators)**.** Let H be a Hilbert space and consider a continuous coercive bilinear form $a \colon H \times H \to \mathbb{R}$ on H, which means that there exist constants $C, \alpha > 0$ such that

$$|a(u,v)| \le C\|u\|_H\|v\|_H \qquad \text{for all } u, v \in H \qquad \text{(continuity/boundedness)}$$

and

$$a(u,u) \ge \alpha\|u\|_H^2 \qquad \text{for all } u \in H \qquad \text{(coercivity).}$$

(a) Prove the world-famous *Lax-Milgram lemma*: For every $f \in H^*$, there exists a unique $u = u_f \in H$ such that

$$a(u,v) = \langle f, v\rangle_{H^*,H} \quad \text{for all } v \in H$$

and there holds $\|u_f\|_H \leq \alpha^{-1}\|f\|_{H^*}$.

Hints:

(i) Recall the also world-famous *Fréchet-Riesz representation theorem*: There is a continuous linear isometric isomorphism $T \in \mathcal{L}(H^*; H)$ such that, for all $g \in H^*$, we have $\langle g, v\rangle_{H^*,H} = (Tg, v)_H$ for all $v \in H$.

(ii) Let $M \subseteq H$. Then $(u,v)_H = 0$ for all $u \in M$ implies $v = 0$ if and only if M is dense in H. (Prove this if needed!)

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\mu \in L^\infty(\Omega; \mathbb{S}_n)$, where $\mathbb{S}_n$ is the set of symmetric real $n \times n$-matrices equipped with the operator-norm inherited from $\|\cdot\|_2$ on $\mathbb{R}^n$.

(i) Show that the *weak divergence-gradient operator* A_μ given by

$$\langle A_\mu u, v\rangle := \int_\Omega \big(\mu\nabla u\big)\cdot\nabla v \;\mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega)$$

for $u \in H_0^1(\Omega)$ is a linear continuous operator $H_0^1(\Omega) \to H^{-1}(\Omega) = H_0^1(\Omega)^*$.

(ii) Suppose that there is $\mu_0 > 0$ such that μ additionally satisfies

$$v^T\mu v \geq \mu_0\|v\|_2^2 \quad \text{for all } v \in \mathbb{R}^n \quad \text{for almost all } x \in \Omega.$$

Show that then for every $f \in H^{-1}(\Omega)$ there is a unique solution $u = u_f \in H_0^1(\Omega)$ of the weak formulation

$$\int_\Omega \big(\mu\nabla u\big)\cdot\nabla v \;\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega)$$

of the elliptic second-order partial differential equation

$$\begin{aligned} -\operatorname{div}\big(\mu\nabla u\big) &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

(This equation is to be seen formally, because μ and f are too general for the equation to be interpreted in a classic sense.) The function $u = u_f$ moreover satisfies $\|u_f\|_{H_0^1(\Omega)} \leq \mu_0^{-1}\|f\|_{H^{-1}(\Omega)}$, so $A_\mu^{-1} \in \mathcal{L}(H^{-1}(\Omega); H_0^1(\Omega))$, and it is also the unique solution of the minimization problem

$$\min_{w \in H_0^1(\Omega)} \frac{1}{2}\int_\Omega \big(\mu\nabla w\big)\cdot\nabla w \;\mathrm{d}x - \langle f, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{1}$$

Hint: Recall that $u \mapsto \|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm on $H_0^1(\Omega)$.

Exercise 3 (Projection formula for the optimal control)**.** Consider a bounded domain $\Omega \subset \mathbb{R}^n$ and the optimal control problem

$$\begin{aligned} \min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} \frac{1}{2}\int_\Omega |y-y_d|^2 \, \mathrm{d}x + \frac{\beta}{2}\int_\Omega |u|^2 \, \mathrm{d}x \\ \text{s.t.} \quad Ay = \mathcal{E}u \ \text{ in } H^{-1}(\Omega) \end{aligned} \tag{Ell-OCP}$$

with $A \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$ and $A^{-1} \in \mathcal{L}(H^{-1}(\Omega); H_0^1(\Omega))$; imagine the divergence-gradient operators from exercise 2. Moreover, $\mathcal{E} \in \mathcal{L}(L^2(\Omega); H^{-1}(\Omega))$ denotes the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and we have $y_d \in L^2(\Omega)$ and $\beta > 0$.

(a) Show that every feasible pair (y, u) is regular.

(b) Let $(\bar{y}, \bar{u})$ be a locally optimal solution of (Ell-OCP). Show that the optimal control $\bar{u}$ is given by
$$\bar{u}(x) = -\beta^{-1}\bar{p}(x) \quad \text{for almost all } x \in \Omega,$$
where $\bar{p} \in H_0^1(\Omega)$ satisfies $A^*\bar{p} = \bar{y} - y_d$. What does this imply for the regularity of $\bar{u}$? What if we can show higher H^2-regularity properties for A and/or A^* as in the example in the lecture notes?

(c) Now assume that there are also control constraints of the form
$$u \in U_{\mathrm{ad}} = \left\{ w \in L^2(\Omega) \colon a \le w \le b \text{ a.e. on } \Omega \right\}$$
in (Ell-OCP), with $L^2(\Omega)$-functions $a \le b$. Show that the optimal control $\bar{u}$ then satisfies
$$\bar{u}(x) = \operatorname{proj}_{[a(x),b(x)]}\left(-\beta^{-1}\bar{p}(x)\right) \quad \text{for almost all } x \in \Omega.$$
Make an educated guess about the regularity of $\bar{u}$ in this case and how an analogous result to the (control-) unconstrained case could be achieved.

Exercise 4 (Partial ordering induced by pointed cone)**.** Let X be a Banach space and let $K \subset X$ be a closed convex and pointed cone, that is, $K \cap (-K) = \{0\}$. Show that the relation $\le_K$ given by
$$x_1 \le_K x_2 \quad \Longleftrightarrow \quad x_2 - x_1 \in -K$$
is a *partial ordering*, that is, it is reflexive, anti-symmetric and transitive. Convince yourself that you are allowed to cancel positive factors α on both sides.