
Exercise 1 (Approximation of the tangential cone)**.** Let $\bar{x} \in \mathcal{F} = G^{-1}[K]$ be regular.

(a) Show that
$$\operatorname{dist}\big(x - \bar{x}, T_\ell(G,K,\bar{x})\big) = o\big(\|x-\bar{x}\|_X\big) \tag{1}$$
for $\mathcal{F} \ni x \to \bar{x}$.

(b) Give an alternative proof of Lemma 3.47, so show that there exists a map $h\colon \mathcal{F} \to T_\ell(G,K,\bar{x})$ with
$$\big\|h(x) - (x-\bar{x})\big\|_X = o\big(\|x-\bar{x}\|_X\big) \quad \text{for } \mathcal{F} \ni x \to \bar{x}.$$

Exercise 2 (Necessary optimality conditions for a simply constrained problem)**.** Let X be a Banach space with $K \subseteq X$ nonempty and convex. Let further $f\colon U \to \mathbb{R}$, where $U \supset K$ is an open set, be twice G-differentiable around the locally optimal solution $\bar{x}$ of the optimization problem
$$\min\, f(x) \quad \text{s.t.} \quad x \in K. \tag{OP}$$

(a) Show that $\bar{x}$ satisfies
$$\big\langle f'(\bar{x}), x - \bar{x}\big\rangle_{X^*,X} \geq 0 \qquad \text{for all } x \in K$$
and
$$f''(\bar{x})\big[x-\bar{x}, x-\bar{x}\big] \geq 0 \qquad \text{for all } x \in K \text{ with } \big\langle f'(\bar{x}), x-\bar{x}\big\rangle_{X^*,X} = 0.$$

(b) Now suppose that $X = L^2(\Omega)$ for some domain $\Omega \subseteq \mathbb{R}^n$ and let
$$K := \big\{w \in L^2(\Omega)\colon a \leq w \leq b\big\},$$
where $a, b \in L^2(\Omega)$ and $a < b$ almost everywhere on Ω. Consider $\nabla f(\bar{x}) \in L^2(\Omega)$, so the representation of $f'(\bar{x}) \in L^2(\Omega)^*$ w.r.t. the $L^2(\Omega)$-scalar product. Find pointwise (almost everywhere) conditions on $\nabla f(\bar{x})$ from the necessary optimality conditions derived in the foregoing part of this exercise.

(c) Derive the KKT-conditions for (OP) and compare them with the pointwise conditions on $\nabla f(\bar{x})$.

Exercise 3. Gotta ~~catch~~ do 'em all! Solve the remaining exercises from the previous exercise sheets.