
Exercise 1 (Approximation of the tangential cone)**.** Let $\bar{x} \in \mathcal{F} = G^{-1}[K]$ be regular.

(a) Show that
$$\operatorname{dist}\big(x - \bar{x}, T_\ell(G,K,\bar{x})\big) = o\big(\|x - \bar{x}\|_X\big) \tag{1}$$
for $\mathcal{F} \ni x \to \bar{x}$.

(b) Give an alternative proof of Lemma 3.47, so show that there exists a map $h\colon \mathcal{F} \to T_\ell(G,K,\bar{x})$ with
$$\big\|h(x) - (x - \bar{x})\big\|_X = o\big(\|x - \bar{x}\|_X\big) \quad \text{for } \mathcal{F} \ni x \to \bar{x}.$$

Solution.

(a) We use Theorem 3.19 by Robinson about metric regularity. A special case of the theorem, cf. Remark 3.20, says that if $\bar{x} \in \mathcal{F}$ is regular for the constraint $G(x) \in K$, then there exist $c, \delta > 0$ such that
$$\operatorname{dist}(x, \mathcal{F}) \le c \operatorname{dist}(G(x), K) \tag{2}$$
for all $x \in B_{\delta,X}(\bar{x})$.

The idea is to use this result for a suitable constraint such that we are able to reproduce (1) with (2). The feasible set $\mathcal{F}$ now needs to be $T_\ell(G,K,G(\bar{x}))$, so we propose the constraint $G'(\bar{x})d \in T(K,G(\bar{x}))$, since $T_\ell(G,K,G(\bar{x}))$ is exactly defined to consist of all those $d \in X$ for which $G'(\bar{x})d \in T(K,G(\bar{x}))$.

As we have to show (1) for $x \to \bar{x}$, the designated regular point of the new constraint is $\bar{d} = 0$. To verify that this is indeed a regular point for the constraint $G'(\bar{x})d \in T(K,G(\bar{x}))$, consider RCQ for this new problem:
$$0 \overset{!}{\in} \operatorname{int}\big\{G'(\bar{x})\bar{d} + G'(\bar{x})X - T(K,G(\bar{x})\big\} = \operatorname{int}\big\{G'(\bar{x})X - \overline{\operatorname{cone}(K,G(\bar{x}))}\big\}.$$

On the other hand, we already know that $\bar{x}$ is regular for the original constraint $G(x) \in K$, hence
$$\begin{aligned} 0 \in \operatorname{int}\big\{G(\bar{x}) + G'(\bar{x})X - K\big\} &= \operatorname{int}\big\{G'(\bar{x})X - \big(K - G(\bar{x})\big)\big\} \\ &\subset \operatorname{int}\big\{G'(\bar{x})X - \overline{\operatorname{cone}(K,G(\bar{x}))}\big\}, \end{aligned}$$

which shows that $\bar{d}$ is indeed regular for the new constraint. Hence, Theorem 3.19 gives us that there exist $c, \delta > 0$ such that
$$\operatorname{dist}(d, T_\ell(G,K,\bar{x})) \le c \operatorname{dist}\big(G'(\bar{x})d, T(K,G(\bar{x}))\big)$$

for all $d \in B_{\delta,X}(0)$. It remains to show that $\operatorname{dist}\big(G'(\bar{x})(x-\bar{x}), T(K, G(\bar{x}))\big) = o(\|x-\bar{x}\|_X)$ for $x \in \mathcal{F}$ close to $\bar{x}$. This follows from observing that

$$\begin{aligned}
\operatorname{dist}\big(G'(\bar{x})d, T(K, G(\bar{x}))\big) &= \inf_{h \in T(K, G(\bar{x}))} \big\|h - G'(\bar{x})d\big\|_X \\
&\leq \inf_{\substack{\lambda \geq 0,\\ h \in K}} \big\|\lambda(h - G(\bar{x})) - G'(\bar{x})d\big\|_X \\
&\leq \inf_{h \in K} \big\|h - G(\bar{x}) - G'(\bar{x})d\big\|_X
\end{aligned}$$

for all $d \in X$, because then the form $d = x - \bar{x}$ with $x \in \mathcal{F}$ and the choice $h = G(x) \in K$ yields

$$\operatorname{dist}\big(G'(\bar{x})d, T(K, G(\bar{x}))\big) \leq \big\|G(x) - G(\bar{x}) - G'(\bar{x})(x-\bar{x})\big\|_X = o\big(\|x-\bar{x}\|_X\big)$$

by F-differentiability of G.

(b) Having (1) at hand, we know that by definition of the distance, for every $x \in \mathcal{F}$ there exists $h = h(x) \in T_\ell(G, K, \bar{x})$ such that

$$\big\|h(x) - (x - \bar{x})\big\|_X \leq \operatorname{dist}\big(x - \bar{x}, T_\ell(G, K, \bar{x})\big) + \|x - \bar{x}\|_X^2.$$

This choice already gives rise to the searched-for map $h\colon \mathcal{F} \to T_\ell(G, K, \bar{x})$ since the right-hand side in the foregoing inequality is already of order $o(\|x-\bar{x}\|_X)$ by (1).

Exercise 2 (Necessary optimality conditions for a simply constrained problem)**.** Let X be a Banach space with $K \subseteq X$ nonempty and convex. Let further $f\colon U \to \mathbb{R}$, where $U \supset K$ is an open set, be twice G-differentiable around the locally optimal solution $\bar{x}$ of the optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in K. \tag{OP}$$

(a) Show that $\bar{x}$ satisfies

$$\big\langle f'(\bar{x}), x - \bar{x}\big\rangle_{X^*,X} \geq 0 \qquad \text{for all } x \in K$$

and

$$f''(\bar{x})\big[x - \bar{x}, x - \bar{x}\big] \geq 0 \qquad \text{for all } x \in K \text{ with } \big\langle f'(\bar{x}), x - \bar{x}\big\rangle_{X^*,X} = 0.$$

(b) Now suppose that $X = L^2(\Omega)$ for some domain $\Omega \subseteq \mathbb{R}^n$ and let

$$K := \big\{w \in L^2(\Omega)\colon a \leq w \leq b\big\},$$

where $a, b \in L^2(\Omega)$ and $a < b$ almost everywhere on Ω. Consider $\nabla f(\bar{x}) \in L^2(\Omega)$, so the representation of $f'(\bar{x}) \in L^2(\Omega)^*$ w.r.t. the $L^2(\Omega)$-scalar product. Find pointwise (almost everywhere) conditions on $\nabla f(\bar{x})$ from the necessary optimality conditions derived in the foregoing part of this exercise.

(c) Derive the KKT-conditions for (OP) and compare them with the pointwise conditions on $\nabla f(\bar{x})$.

Solution.

(a) The proofs work exactly as in Nonlinear Optimization. Since we know that $\bar{x}$ is locally optimal for (OP), that K is convex and that f is G-differentiable around $\bar{x}$, we find

$$0 \le \frac{f\big(\bar{x} + t(x - \bar{x})\big) - f(\bar{x})}{t} \quad \text{for all } x \in K,\ t \in [0,1],$$

hence

$$0 \le \lim_{t \searrow 0} \frac{f\big(\bar{x} + t(x - \bar{x})\big) - f(\bar{x})}{t} = \big\langle f'(\bar{x}), x - \bar{x} \big\rangle_{X^*,X} \quad \text{for all } x \in K.$$

For the second assertion, consider the Taylor expansion of $t \mapsto f(\bar{x} + t(x - \bar{x}))$ and observe that

$$0 \le f\big(\bar{x} + t(x - \bar{x})\big) - f(\bar{x}) = t\big\langle f'(\bar{x}), x - \bar{x} \big\rangle_{X^*,X} + \frac{t^2}{2} f''(\bar{x})\big[x - \bar{x}, x - \bar{x}\big] + o(t^2)$$

as $t \searrow 0$. If now $\big\langle f'(\bar{x}), x - \bar{x} \big\rangle_{X^*,X}$, then we obtain

$$f''(\bar{x})\big[x - \bar{x}, x - \bar{x}\big] \ge \frac{o(t^2)}{t^2} \to 0 \quad \text{as } t \searrow 0.$$

(b) The gradient $\nabla f(\bar{x}) \in L^2(\Omega)$ is defined to be precisely the function which satisfies

$$\big\langle f'(\bar{x}), h \big\rangle_{L^2(\Omega)^*, L^2(\Omega)} = \big(\nabla f(\bar{x}), h\big)_{L^2(\Omega)} = \int_\Omega \nabla f(\bar{x})(h)\,\mathrm{d}t \quad \text{for all } h \in L^2(\Omega),$$

so in particular

$$\begin{aligned} \big\langle f'(\bar{x}), x - \bar{x} \big\rangle_{L^2(\Omega)^*, L^2(\Omega)} &= \big(\nabla f(\bar{x}), x - \bar{x}\big)_{L^2(\Omega)} \\ &= \int_\Omega \nabla f(\bar{x})(x - \bar{x})\,\mathrm{d}t \quad \text{for all } x \in K. \end{aligned}$$

So, if $\bar{x}$ is locally optimal for (OP), then we have

$$\int_\Omega \nabla f(\bar{x})(x - \bar{x})\,\mathrm{d}t \ge 0 \quad \text{for all } x \in K.$$

This allows to derive the following conditions on $\nabla f(\bar{x})$: Consider the set

$$\Omega(b) := \big\{ t \in \Omega \colon \bar{x}(t) = b(t) \big\}.$$

We know that $x(t) - \bar{x}(t) \leq 0$ for almost all $t \in \Omega(b)$ and all $x \in K$. Now assume that $\nabla f(\bar{x})(t) > 0$ for some set $\Xi \subseteq \Omega(b)$ with nonzero Lebesgue measure and consider the function $x = \chi_\Xi \cdot (a - \bar{x}) + \bar{x} \in K$. Then

$$\int_\Omega \nabla f(\bar{x})(x - \bar{x})\,\mathrm{d}t = \int_\Xi \underbrace{\nabla f(\bar{x})}_{>0} \cdot \underbrace{(a - b)}_{>0}\,\mathrm{d}t < 0,$$

which is a contradiction. Hence, $\nabla f(\bar{x}) \leq 0$ almost everywhere on $\Omega(b)$. Analogously, one shows that $\nabla f(\bar{x}) \geq 0$ almost everywhere on $\Omega(a)$. Finally, $\nabla f(\bar{x})$ must be zero almost everywhere on the set $\Omega(a, b)$ where $\bar{x}(t) \in (a(t), b(t))$ as one sees immediately by considering the functions $x = \chi_\Xi(a - \bar{x}) + \bar{x}$ and $x = \chi_\Xi(b - \bar{x}) + \bar{x}$ with $\Xi \subseteq \Omega(a, b)$ having nonzero Lebesgue measure.

Altogether, we arrive at

$$\nabla f(\bar{x})(t) \begin{cases} \leq 0 & \text{if } \bar{x}(t) = b(t), \\ \geq 0 & \text{if } \bar{x}(t) = a(t), \\ = 0 & \text{otherwise} \end{cases} \quad \text{for almost every } t \in \Omega.$$

(c) The KKT conditions for (OP) are given by

$$f'(\bar{x}) + \bar{\lambda} = 0 \quad \text{in } L^2(\Omega)^* \quad \text{and} \quad \bar{\lambda} \in T(K, \bar{x})^\circ.$$

The latter implies that $\langle \bar{\lambda}, d \rangle \leq 0$ for all $d \in \operatorname{cone}(K, \bar{x})$, so in particular

$$\left\langle f'(\bar{x}), x - \bar{x} \right\rangle_{L^2(\Omega)^*, L^2(\Omega)} = \left\langle -\bar{\lambda}, x - \bar{x} \right\rangle_{L^2(\Omega)^*, L^2(\Omega)} \geq 0 \quad \text{for all } x \in K.$$

From here, one argues as above. The KKT conditions thus yield the same necessary pointwise representation of $\nabla f(\bar{x})$ as the "basic" necessary first-order optimality conditions.

Exercise 3. Gotta ~~catch~~ do 'em all! Solve the remaining exercises from the previous exercise sheets.