The full Keller-Segel model is well-posed on nonsmooth domains

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Abstract. In this paper we prove that the full Keller-Segel system, a quasilinear strongly coupled reaction-crossdiffusion system of four parabolic equations, is well-posed in space dimensions 2 and 3 in the sense that it always admits an unique local-in-time solution in an adequate function space, provided that the initial values are suitably regular. The proof is done via an abstract solution theorem for nonlocal quasilinear equations by Amann and is carried out for general source terms. It is fundamentally based on recent nontrivial elliptic and parabolic regularity results which hold true even on rather general nonsmooth spatial domains. This enables us to work in a nonsmooth setting which is not available in classical parabolic systems theory. Apparently, there exists no comparable existence result for the full Keller-Segel system up to now. Due to the large class of possibly nonsmooth domains admitted, we also obtain new results for the "standard" Keller-Segel system consisting of only two equations as a special case.

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1. Introduction

This paper establishes the local-in-time existence of solutions in a suitable functional-analytic sense to the so-called original *full Keller-Segel model* which is a coupled system of four nonlinear parabolic partial differential equations over a finite time horizon $J =]0,T[$ in a bounded domain $\Omega \subset \mathbb{R}^d$ in space dimensions $d \in \{2,3\}$, and reads as follows:

$$u' - \operatorname{div}(\kappa(u,v)\nabla u) = \operatorname{div}(\sigma(u,v)\nabla v) \qquad \text{in } J \times \Omega, \tag{1.1}$$

$$v' - k_v \Delta v = -r_1 vp + r_{-1} w + uf(v) \qquad \text{in } J \times \Omega, \tag{1.2}$$

$$p' - k_p \Delta p = -r_1 vp + (r_{-1} + r_2) w + ug(v,p) \qquad \text{in } J \times \Omega, \tag{1.3}$$

$$w' - k_w \Delta w = r_1 vp - (r_{-1} + r_2) w \qquad \text{in } J \times \Omega, \tag{1.4}$$

combined with homogeneous Neumann conditions

$$\nu \cdot \kappa(u,v)\nabla u = \nu \cdot k_v \nabla v = \nu \cdot k_p \nabla p = \nu \cdot k_w \nabla w = 0 \qquad \text{on } J \times \partial\Omega, \tag{1.5}$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$, and suitable initial values

$$\big(u(0,\cdot), v(0,\cdot), p(0,\cdot), w(0,\cdot)\big) = (u_0, v_0, p_0, w_0) \qquad \text{in } \Omega. \tag{1.6}$$

Before we elaborate on the origin and biological meaning of this model, let us explain a critical property of this system of parabolic equations. The coefficient function σ in (1.1) is *not* assumed to be definite in sign and generally not restricted in its magnitude. This implies that the spatial second order system differential operator underlying (1.1)–(1.4) fails to satisfy the usual strong ellipticity conditions in the form of the Legendre– or Legendre-Hadamard conditions; in particular, there is in general no Gårding inequality available, cf. [35, 99]. The system *is* normally elliptic in the sense of Amann [3]—also known as Petrowskii parabolic [63, Ch. VII.8]—and as such admits existence of local-in-time solutions quite immediately under the assumptions there. These assumptions however include "smoothness", or at least C^2-regularity, of the boundary $\partial\Omega$ and it is not clear how to adopt the theory to less smooth situations. Under such smoothness assumptions, the results in [3] have been used already to obtain local-in-time existence of a related system, cf. (1.7) below, for instance in [19, 87, 95]. Let us note that the authors in [34] deal with a related system in a piecewise C^2-setting.

It is the aim of this work to show the existence of local-in-time solutions of (1.1)–(1.6) in a generally *nonsmooth* setting for Ω, namely that of a Lipschitz domain. Since we, as explained, cannot use established theory for parabolic systems, the strategy for our proof is to solve the lower three equations for (v,p,w) in dependence of the function u and to re-insert this dependence for v in the first equation. This way, we obtain a *single*, albeit quite involved, parabolic equation for u for which we can rely on the full power of recent elliptic ([10, 28, 41]) and parabolic ([38, 45]) results, available for very general geometric constellations, in order to treat it, thereby using a fundamental theorem by Amann [4, Thm. 2.1]. Following this strategy, we also obtain new results for the related system mentioned above, the classical two-equation Keller-Segel model of chemotaxis ((1.7) below), and similar systems in a nonsmooth setting.

The consideration of a nonsmooth boundary for Ω is not an academic example but motivated by observations from numerical simulations of both (1.1)–(1.6) and simplified models. For instance, these numerical simulations show a concentration behavior of the solution in the smallest interior angle of the considered domain. There is also a connection between the geometry of the domain and the precise critical mass that insures the global-in-time existence of a solution on nonsmooth domains, given as a multiple of the smallest interior angle of the domain (see for example [34, Thm. 4.3, Rem. 4.5]). In this sense, it is of interest to establish (local-in-time) existence results also for a generally nonsmooth boundary of Ω.

1.1. Biological background

The above model describes the aggregation phase during the life cycle of cellular slime molds like the *Dictyostelium discoideum* and was first introduced by Keller and Segel in their 1970ies paper [59]. We briefly describe the underlying biological processes. Looking at its life cycle one observes that a myxamoebae population of the Dictyostelium grows by cell division as long as there are enough food resources. When these are depleted, the myxamoebae propagate over the entire domain available to them. Then, after a while, the phase that is covered by the given model is initiated by one cell that starts to exude cyclic Adenosine Monophosphate (cAMP) which attracts the other myxamoebae. As a consequence the other myxamoebae are stimulated to move in direction of the so-called founder cell and commence to release cAMP. This leads to the aggregation of the myxamoebae that also start to differentiate within the myxamoebae aggregates resp. within the aggregation centers. The aggregation phase ends with the formation of a pseudoplasmoid in which every myxamoebae maintains its individual integrity. However, Keller and Segel did not model the formation of the pseudoplamoid; thus, this phase of the life cycle of the Dictyostelium is *not* covered in the original equations. This pseudoplasmoid is attracted by light and, therefore, it moves towards light sources. Finally a fruiting body is formed and after some time spores are diffused from which the life cycle begins again. For more details on the life cycle of the Dictyostelium we refer to [15], for example.

In the given model $u(t,x)$ denotes the myxamoebae density of the cellular slime molds at time t in point x, where $v(t,x)$ describes a chemo-attractant concentration (like cAMP). The given model for aggregation of a cellular slime population is based on four basic processes that can be observed during the aggregation phase:

a) The chemo-attractant is produced per amoeba at a positive rate $f(v)$.

b) The chemo-attractant is degraded by an extra-cellular enzyme, where the concentration of the is enzyme at time t in point x is denoted by $p(t,x)$. This enzyme is produced by the myxamoebae at a positive rate $g(v,p)$ per amoeba.

c) Following Michaelis-Menten the chemo-attractant and the enzyme react to form a complex $\mathcal{E}$ of concentration w which dissociates into a free enzyme plus the degraded product:

$$v + p \quad \underset{r_{-1}}{\overset{r_1}{\rightleftarrows}} \quad \mathcal{E} \quad \xrightarrow{r_2} \quad p + \text{ degraded product,}$$

where r_{-1}, r_1 and r_2 are positive constants representing the reaction rates.

d) The chemo-attractant, the enzyme and the complex diffuse according to Fick's law.

As a tribute to the experimental setting and the conservation of the myxamoebae density the equations are equipped with homogeneous Neumann boundary data.

Since the influence of chemical substances in the environment on the movement of motile species (in general called chemotaxis) can lead to strictly oriented or to partially oriented and partially tumbling movement of the species, the first equation contains both a pure diffusion term $\operatorname{div}(\kappa(u,v)\nabla u)$ with $\kappa(u,v) \geq 0$ for nonnegative functions (u,v), and a convection term $\operatorname{div}(\sigma(u,v)\nabla v)$ which describes the movement with respect to the chemical concentration. For a movement towards a higher concentration of the chemical substance, termed *positive* chemotaxis, one assumes $\sigma(u,v) < 0$ for nonnegative (u,v), while for the movement towards regions of lower chemical concentration, called *negative* chemotactical movement, the opposite inequality $\sigma(u,v) > 0$ has to hold. For the detailed derivation of the given model we refer to [51, 59].

Chemotaxis is known to be an important device for cellular communication. In development or in living tissues the communication by chemical signals prearranges how cells collocate and organize themselves. Biologists studying chemotaxis often concentrate their experiments on the movement, the self-organization and pattern formations of the cellular slime mold Dictyostelium discoideum. One reason for the great interest in this cellular slime mold is caused by the fact that "*development in Dictyostelium discoideum results only in two terminal cell types, but processes of morphogenesis and pattern formation occur as in many higher organisms*" (see [76, p. 354]). Thus biologists hope that studying this cellular slime mold gives more insights in understanding cell differentiation.

1.2. Context and related work

By to a simplification done by Keller and Segel themselves in [59], the original model of four strongly coupled parabolic equations (1.1)–(1.4) was reduced to a model which is given by a system of only *two* strongly coupled parabolic equations. This was done by assuming that the complex is in a steady state with regard to the chemical reaction and that the total concentration of the free and the bounded enzyme is a constant; assumptions that are well-known for the Michaelis-Menten equations in enzyme kinetics. The reduction was justified by the paradigm that "*it is useful for the sake of clarity to employ the simplest reasonable model*" (see [59, p. 403]). The corresponding model is then given by the following parabolic equations:

$$\left.\begin{aligned} u' - \operatorname{div}(\kappa(u,v)\nabla u) &= \operatorname{div}(\sigma(u,v)\nabla v) && \text{in } J \times \Omega, \\ v_t - k_c \Delta v &= -k(v)v + uf(v) && \text{in } J \times \Omega, \\ \nu \cdot \kappa(u,v)\nabla u = \nu \cdot k_c \nabla v &= 0 && \text{on } J \times \partial\Omega, \\ \big(u(0,\cdot), v(0,\cdot)\big) &= (u_0, v_0) && \text{in } \Omega. \end{aligned}\right\} \tag{1.7}$$

This model is nowadays often referred to as the classical chemotaxis model or as the Keller-Segel model in chemotaxis. As in the full model, $\kappa(u,v)$ denotes the density-dependent diffusion coefficient and $\sigma(u,v)$ is the chemotactic sensitivity, where now $k(v)v$ and $uf(v)$ describe degradation and production of the chemical signal. For $\kappa(u,v) = 1$, $\sigma(u,v) = -\chi \cdot u$ or $-\chi \frac{u}{v}$ with a constant $\chi > 0$ and $k(\cdot)$ and $f(\cdot)$ positive constants, this two-equation model has been extensively studied during the last twenty years, see for instance [46, 47, 51, 52, 55] and the references therein. In particular the so-called Childress-Percus conjecture [17] for (1.7) concerning L^∞ blow-up behavior has attracted many scientists. Subdividing via space dimension we mention [77] for $d = 1$ and, among others, [13, 34, 43, 48, 49, 50, 74] for $d = 2$, as well as [16, 20, 54, 56, 93] for $d = 3$.

From the biological point of view, the blow-up behavior of the solution can be interpreted as the starting point of cell differentiation and therefore the blow-up time $T_{\max} < \infty$ would correspond to the stopping time where the aggregation phase in the life cycle of the Dictyostelium ends and the cell differentiation and formation of the pseudoplasmoid starts.

Besides the mathematical interesting question whether the solution can blow up in finite or in infinite time one can also observe interesting pattern formations during the aggregation phase and development of the Dictyostelium such as traveling waves like motion and spiral waves for the chemo-attractant. Although there have been some attempts to prove the existence of traveling wave solutions and to simulate sunflower spirals for the simplified model (1.7)—see for instance [8, 52, 53, 91] and the references therein—, one seems to need more complicated chemotaxis systems consisting of more than only two equations to describe such kind of pattern formation. However, these more complicated systems still fit in the general setting of the full Keller-Segel model as considered in the present paper (cf. (4.1)–(4.5) on page 18 below). Hence, it might be worthwhile to work on the original four-equation-system instead if one tries to describe these pattern formations during the aggregation of some particular species. Possibly, the reduction to two equations that was done in [59] was too restrictive to cover all observable patterns and phenomena during the aggregation of mobile species like the Dictyostelium discoideum. As another example, one can find an attempt to describe the aggregation of the Dictyostelium discoideum along the experimentally observable cAMP spiral waves in [90] where the authors consider a coupled three-equations model that contains a version of the simplified Keller-Segel model complemented with an ODE that covers the recovery process of the myxamoebae after binding the extra-cellular cAMP. As above, it seems worthwhile to investigate the original full model to see whether it can also generate these complex pattern formations.

As far as we know there are no results available for the full four-equation model on nonsmooth domains. In particular, the question of blow-up has, as far as we know, not been studied for the full four equations model up to now. Of course, there are several local-existence results known for parabolic-parabolic and parabolic-elliptic versions of the simplified two equation model (1.7) as for instance the results in [2, 13, 14, 46, 73, 85, 86, 97]. Furthermore, existence results for solutions for the simplified two-equation model with additional population growth are also known, cf. [58, 78, 88, 92, 96]. Some of these results may be extended to the full model (1.1)–(1.6); however, all of them consider the equation either on a smooth domain with boundary of class C^2, on convex domains with smooth boundaries, or on the whole space $\mathbb{R}^d$. Furthermore, the initial data has to satisfy certain comparability conditions in some cases. The only result which we are aware of concerning nonsmooth objects is the local existence result in [34] where the authors allow a domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ that is piecewise of class C^2. It will moreover turn out that the analysis presented below for the full model (1.1)–(1.6) immediately transfers to the more simple model (1.7). Therefore, the results stated in the present paper are completely new and much more general than those known so far.

1.3. Outline and strategy

Our analysis of the system (1.1)–(1.4) fundamentally bases on the fact that it is only *one* equation, (1.1), where the second derivative of another quantity appears. So we solve the equations (1.2)–(1.4) for (v, p, w), where u enters parametrically as a given function. It turns out that the dependence of (v, p, w) on u in this spirit is well-behaved in a suitable sense. This allows to insert (v, p, w) in their dependence of u into (1.1). Thus, one ends up with *one* "scalar" quasilinear parabolic equation whose dependence on u is *nonlocal in time*, since the functions v, p, w, as solutions to evolution equations themselves, depend on the *whole* function u on $[0, t]$ instead of just the value $u(t)$. Such an equation, however, can be solved by a pioneering theorem of Amann which covers such general settings, cf. [4, Thm. 2.1] or Theorem 3.17 below. Still, it is a formidable task to verify the assumptions of the theorem, since the equation under consideration is still quasilinear and nonlocal in nature.

Thereby it is not obvious a priori in which function spaces the problem should be considered, but since *homogeneous* Neumann conditions are prescribed, cf. (1.5), Lebesgue spaces $L^p(\Omega)$ are a favorable choice since the boundary conditions are reflected in a strong sense by the differential operators there, see Remark 3.3 below. Fortunately, there are various recent elliptic ([10, 28, 41]) and parabolic ([38, 45]) regularity results available which are even valid in the case of non-smooth domains and which allow for a treatment of (1.1)–(1.4) in this setting. The indeed crucial problem is the adequate choice of the integrability order p. However, there is fairly general class of domains Ω for which the divergence-gradient operator $-\nabla \cdot \mu \nabla$ admits maximal Sobolev regularity on $W^{1,q}$ for some $q > d$, that is,

$$-\nabla \cdot \mu \nabla + 1 \colon \quad W^{1,q}(\Omega) \to \big(W^{1,q'}(\Omega)\big)' =: W_{\bullet}^{-1,q}(\Omega) \tag{1.8}$$

is a topological isomorphism, where μ is a bounded, measurable and strictly positive function on Ω, cf. [28, 41] (see Chapter 3 for precise definitions). Combining this isomorphism property with recent and powerful results on the square root of elliptic operators as in [10, Thm. 5.1] (see also Proposition 3.8 below) provides very precise embedding results for the domains of fractional powers of the elliptic operators on Lebesgue spaces $L^p(\Omega)$. On the other hand, one can show that the domains of the operators $-\nabla \cdot \phi \mu \nabla$, when considered on $L^{\frac{q}{2}}(\Omega)$, are independent of ϕ, whenever ϕ is a strictly positive function from $W^{1,q}(\Omega)$, cf. e.g. [68] (see also Lemma 3.22 below). This is a crucial property in the task of establishing *constant* domains for the operators entering in the quasilinear equation (1.1), the latter being a central point in the theorem of Amann mentioned above, for which we then indeed choose a Lebesgue space $L^p(\Omega)$ with $p = \frac{q}{2}$ for $q > d$ satisfying (1.8).

Note that for the Keller-Segel model (1.1)–(1.6) one in fact only needs to consider $\mu \equiv 1$, but our technique is not necessarily restricted to the Laplacian or even only scalar multipliers within the divergence-gradient operator, cf. our comments in Chapter 5 at the end of the paper.

Let us emphasize that this strategy for the analysis of the system (1.1)–(1.6) may be adopted to both the simplified model (1.7) and the situation where the equations (1.2)–(1.4) for v, p, and w are *elliptic* only, with virtually no changes. For the latter case, one would even have an immediate relation between $(v(t), p(t), w(t))$ and $u(t)$ for each $t \in J$, i.e., a local dependency of (v, t, p) on u, for which the resulting reduced equation for u is then tractable using the slightly less restrictive theorem of Prüss [83] instead of the result of Amann suitable for nonlocal dependencies.

See [69] for a display of this technique where the (single) elliptic equation is even also quasilinear.

The outline of the paper is as follows: in the next chapter we will establish notations, general assumptions and definitions. In Chapter 3, we collect preliminary results, partly already established in other papers. In particular, the concept of maximal parabolic regularity is introduced − being fundamental for all what follows. The investigation of the model is carried out in Chapter 4, beginning with a precise formulation in Chapter 4.1. The main result, local-in-time existence and uniqueness for the Keller-Segel system, is formulated in Theorem 4.3. It follows the proof of this in Chapter 4.2. The paper finishes with concluding comments and remarks in Chapter 5.

2. Notations, general assumptions and definitions

The underlying spatial set Ω is always supposed to be a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$ in the sense of [39, Def. 1.2.1.2] or [67, Ch. 1.1.9]. The reader should carefully notice that this is different from a *strong Lipschitz domain*, which is more restrictive and in fact identical with a *uniform cone domain*, see again [39, Def. 1.2.1.1] or [67, Ch. 1.1.9]. We note that a Lipschitz domain has the extension property, see e.g. [36, Thm. 7.25], such that the usual function space embeddings are available.

Concerning function space terminology, $W^{1,q}(\Omega)$ for $q \in]1,\infty[$ stands for the usual Sobolev space on Ω as a complex vector space (we will switch to real ones later). Accordingly, $W^{-1,q}_{\bullet}(\Omega)$ denotes the *anti*-dual of $W^{1,q'}(\Omega)$. Moreover, for $\theta \in]0,1[$ and $q \in]1,\infty[$, $H^{\theta,q}(\Omega)$ is the symbol for the space of Bessel potentials on Ω, cf. [89, Ch. 4.2.1]. The space of uniformly continuous functions on Ω is denoted by $C(\overline{\Omega})$. For an open set $\Lambda \subset \mathbb{R}^N$, where $N \in \{1,2,3\}$, and a Banach space X, we write $C^\alpha(\Lambda; X)$ for the usual X-valued Hölder spaces of order $\alpha \in]0,1[$, cf. [7, Ch. II.1.1.]. We will mostly encounter these in the incarnations $\Lambda = \Omega$ and $X = \mathbb{R}$ or Λ an interval in $\mathbb{R}$ and X a function space. Since we frequently work with triplets of functions, let $\mathbb{L}^p(\Omega)$ and $\mathbb{W}^{1,q}(\Omega)$ denote the spaces $(L^p(\Omega))^3$ and $(W^{1,q}(\Omega))^3$, respectively. The domain Ω under consideration will not change throughout this work, hence we usually omit the reference to Ω when working with the function spaces.

For two Banach spaces X and Y we denote the space of linear, bounded operators from X into Y by $\mathcal{L}(X;Y)$ with $\mathcal{L}(X) := \mathcal{L}(X;X)$. The norm in a Banach space X will be always indicated by $\|\cdot\|_X$. If a Banach space Y is contained in another Banach space X and the canonical injection of Y into X is continuous, then we say that Y is *embedded* into X and write $Y \hookrightarrow X$. Let Y embed into X. Then $\mathcal{E}(Y;X)$ denotes the *embedding constant*, i.e., the norm of the embedding map. Moreover, in the same situation, if B is the restriction of an operator $A\colon X \supseteq \mathrm{dom}(A) \to X$ to the space Y, then $\mathrm{dom}_Y(B)$ indicates the domain of this operator B in Y.

Finally, we use $J =]0,T[$ for $0 < T < \infty$, and the letter c denotes a generic constant, not always of the same value.

2.1. Assumptions

In order to allow for concise notation in the later stages of this work, we generalize the nonlinear growth, production and degradation terms on the right hand sides of (1.2)–(1.4) to general functions R_2, R_3, R_4, including a function R_1 for (1.1) which is not present in the above model but poses no problem to include analytically. Note that the differential operator for v in (1.1) will be treated specially. For the R_i and for the coefficient functions κ and σ, we make the following assumptions.

Assumption 2.1. i) The functions $\kappa, \sigma \colon \mathbb{R}^2 \to \mathbb{R}$ are supposed to be twice continuously differentiable. Moreover, κ takes only positive values.

ii) For $i = 1, \dots, 4$, each function R_i is defined on $\mathbb{R}^4$ and maps into $\mathbb{R}$, and is also assumed to be twice continuously differentiable.

We point out that we have to pose another assumption of completely different nature than the above ones concerning the regularity of the domain Ω, cf. Assumption 3.6 below. This assumption is only posed below to put it in the appropriate context.

Remark 2.2. In the sequel, the functions κ, σ are always readily identified with the induced superposition operators, acting from $C(\overline{\Omega}) \times C(\overline{\Omega})$ into $C(\overline{\Omega})$. The same is, *mutatis mutandis*, done for the functions R_1, R_2, R_3, R_4.

3. Preliminaries: Some operator theoretic results

In this chapter we declare suitable Banach spaces on which the Keller-Segel system will be considered and in which the analysis is carried out, and the corresponding differential operators. As already explained in the introduction, we plan to treat the system in the L^p scale. Unfortunately, in view of the nonlinearities in the system, the Hilbert space L^2 is *not* appropriate in general, cf. also our comments in Chapter 5 below. It will become clear that L^p-spaces with suitably chosen p, possibly smaller than 2, allow for a suitable treatment of the Keller-Segel system. Thus, it is the aim of the following considerations to provide a consistent definition of the second order divergence operators on such L^p spaces and to show that these operators indeed possess suitable functional analytic properties, in particular, maximal parabolic regularity.

Definition 3.1. Assume that μ is a real-valued, measurable, bounded function on Ω. We define the continuous linear operator

$$-\nabla \cdot \mu \nabla \colon W^{1,2} \to W^{-1,2}_{\bullet}$$

by

$$\left\langle -\nabla \cdot \mu \nabla v, w \right\rangle := \int_\Omega \mu \nabla v \cdot \nabla \overline{w} \, \mathrm{d}x \quad \text{for} \qquad v, w \in W^{1,2}. \tag{3.1}$$

It is convenient to view this operator equivalently as a closed one on $W^{-1,2}_{\bullet}$ with domain $W^{1,2}$. For $q > 2$, we define the operator in $W^{-1,q}_{\bullet}$ by taking the maximal corestriction to that space, thus obtaining again a closed operator, denoted by the same symbols, with a generally unknown domain of definition $\mathrm{dom}_{W^{-1,q}_{\bullet}}(-\nabla \cdot \mu \nabla)$.

Taking $\mu \equiv 1$ in Definition 3.1, one, of course, recovers the (negative) weak Laplacian.

Remark 3.2. In this context, it is not quite common to admit functions μ which take positive *and* negative values. Nevertheless, this is unavoidable by the properties of the function σ originating from the model, cf. the introduction, see also [34].

3.1. The restriction of $-\nabla \cdot \mu\nabla$ to L^p spaces

Let us in this section consider $-\nabla \cdot \mu\nabla$ as in Definition 3.1 as an operator mapping $W^{1,2}$ to $W_\bullet^{-1,2}$. For $p \in [1,\infty[$, we define the *restriction* $A_p(\mu)$ of $-\nabla\cdot\mu\nabla$ to the space L^p as follows: $\psi \in W^{1,2} \cap L^p$ belongs to $\mathrm{dom}_{L^p}(A_p(\mu))$ iff the (anti-) linear form

$$\big(W^{1,2} \cap L^{p'}\big) \ni \varphi \mapsto \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,\mathrm{dx} = \big\langle -\nabla \cdot \mu\nabla\psi, \varphi\big\rangle \tag{3.2}$$

is continuous if $W^{1,2} \cap L^{p'}$ is only equipped with the weaker $L^{p'}$ topology, i.e., if there exists a constant $c = c(\psi)$ such that

$$\big|\langle -\nabla \cdot \mu\nabla\psi, \varphi\rangle\big| \le c(\psi)\|\varphi\|_{L^{p'}} \quad \text{for all } \varphi \in W^{1,2} \cap L^{p'}.$$

In this case, the functional (3.2) may be extended by continuity from the dense subspace $W^{1,2} \cap L^{p'}$ to whole $L^{p'}$ under preservation of its norm. We denote the representative of this functional on $L^{p'}$ by $\Psi \in L^p$ and define $A_p(\mu)\psi := \Psi$. Then $A_p(\mu)\psi$ satisfies

$$\int_\Omega \big(A_p(\mu)\psi\big)\,\overline{\varphi}\,\mathrm{dx} = \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,\mathrm{dx} = \big\langle -\nabla \cdot \mu\nabla\psi, \varphi\big\rangle \quad \text{for all } \varphi \in W^{1,2} \cap L^{p'}, \tag{3.3}$$

which is considered as the *constitutive relation* between $-\nabla \cdot \mu\nabla\psi$ and $A_p(\mu)\psi$. In fact, (3.3) precisely means that $-\nabla\cdot\mu\nabla\psi \in W_\bullet^{-1,2}$ is the image of $A_p(\mu)\psi \in L^p$ under the embedding $L^p \hookrightarrow W_\bullet^{-1,2}$. Moreover, it is clear that the L^p-norm of $A_p(\mu)\psi$ is nothing else but the norm of the antilinear form (3.2) where $W^{1,2} \cap L^{p'}$ is equipped with the $L^{p'}$-norm.

Since the notation $A_p(\mu)$ already indicates the space on which the operator is assumed to act, we write $\mathrm{dom}(A_p(\mu))$ instead of $\mathrm{dom}_{L^p}(A_p(\mu))$ if there is no need for greater care. Note that the often used technique to construct the "strong" differential operators on the L^p scale by restricting $A_2(\mu)$ to L^p for $p > 2$ and taking adjoints of these resulting operators to define the corresponding operator in L^p for $p < 2$ (or forming the closure of $A_2(\mu)$ there) gives the same operators as the procedure above.

We will mostly consider the case of strictly positive μ; only in Lemma 3.22 properties of the operators $A_p(\mu)$ with possibly nonpositive values for μ are pointed out which are fundamental for the treatment of the divergence operator in the right hand side of (1.1). Hence, let us now assume for the rest of this subchapter that μ is bounded from below by a positive constant.

Remark 3.3. It is well-known that the property $\psi \in \mathrm{dom}(A_2(\mu))$ implies a (generalized) homogeneous Neumann condition $\nu \cdot \mu\nabla\psi = \nu \cdot \nabla\psi = 0$ on $\partial\Omega$, cf. [18, Ch. 1.2] or [33, Ch. II.2], ν being the outer normal at the boundary. This fact reflects the homogeneous Neumann boundary conditions (1.5) on the functional analytic level.

We collect some properties of the operators $A_p(\mu)$ and its relation with $-\nabla \cdot \mu\nabla$.

Proposition 3.4. *Let $\mu \in L^\infty(\Omega)$ be a real function with a strictly positive lower bound. Then the Lipschitz property of Ω implies the following assertions:*

i) The operator $A_2(\mu)$ is a non-negative, selfadjoint operator on L^2, classically considered as the operator induced by the form (3.1) *on $W^{1,2}$.*

ii) Under the Lipschitz assumption on Ω, the operators $\nabla \cdot \mu\nabla$ generate analytic semigroups on $W_\bullet^{-1,q}$ for all $q \in [2,\infty[$.

iii) $-A_2(\mu)$ *generates a contractive semigroup* $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ *on* L^2 *which extends consistently to all* L^p *spaces for* $p \in [1,\infty]$ *and is moreover analytic if* $p < \infty$. *These semigroups are also consistent with the ones generated by* $\nabla \cdot \mu\nabla$ *on* $W_\bullet^{-1,q}$ *and their generators are exactly the operators* $-A_p(\mu)$. *The semigroups* $\exp(-t(A_p(\mu)+1))_{t\geq 0}$ *transform* real *functions into* real *ones and* positive *ones into* positive *ones.*

iv) *Both* $-\nabla \cdot \mu\nabla + 1$ *on* $W_\bullet^{-1,q}$ *for* $q \in [2,\infty[$ *and* $A_p(\mu)+1$ *on* L^p *for* $p \in \,]1,\infty[$ *are positive operators; in particular, their fractional powers are well-defined. The operators* $-A_p(\mu)+1$ *even admit bounded imaginary powers: the set of operators* $\{(A_p(\mu)+1)^{\mathrm{i}s} \colon s \in \,]-\varepsilon,\varepsilon[\}$ *is bounded in* $\mathcal{L}(L^p)$ *for every* $p \in \,]1,\infty[$ *and every* $\varepsilon > 0$.

v) *The operator* $A_2(\mu)+1$ *satisfies the Kato square root property, that is, we have* $\mathrm{dom}\big((A_2(\mu)+1)^{\frac12}\big) = W^{1,2}$, *or equivalently,* $(A_2(\mu)+1)^{\frac12}$ *is a topological isomorphism between* $W^{1,2}$ *and* L^2.

Proof. i) See [80, Ch. 1.2.3] or the classical text [61, Ch. VI.2].

ii) See [27, Lem. 6.9(c)].

iii) The extension of $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ to L^p is proven in [80, Corollaries 2.16 and 4.10]. Consistency of the L^p semigroups is shown in [80, Ch. 1.4.2], whereas consistency with the $W_\bullet^{-1,q}$-scale can be found in [31, Ch. 4]. That $-A_p(\mu)$ is the generator of the L^p semigroups follows from the constitutive relation (3.3) and [31, Prop. 2.5]. The mapping properties for real and positive functions are from [80, Ch. 2.6].

iv) The positive operator property for the $W_\bullet^{-1,q}$ operators follows from the same property for the L^p operators, cf. [10, Thm. 11.5], which then implies well-definedness of their fractional powers by [89, Ch. 1.15]. For the bounded imaginary powers, see [22] or [80, Cor. 7.24].

v) This is the classical result of Kato [60, Ch. 5] in conjunction with $A_2(\mu)$ being selfadjoint. □

Remark 3.5. The domain of the operator $A_p(\mu)$ is always equipped with the usual norm $\|(A_p(\mu)+1)\cdot\|_{L^p}$, or $\|(A_p(\mu)+1)\cdot\|_{\mathbb{L}^p}$ when considered on the space L^p or $\mathbb{L}^p$, respectively. This means that $\mathrm{dom}\, A_p(\mu)$ and $\mathrm{dom}\big(A_p(\mu)+1\big)$ coincide as Banach spaces and we will use them interchangeably.

Observing that the fractional powers of $-\nabla\cdot\mu\nabla + 1$ and $A_p(\mu)+1$ are well-defined, the boundedness of the imaginary powers of $A_p(\mu)+1$ in particular implies the identity of the domains of fractional powers $(A_p(\mu)+1)^\alpha$ with interpolation spaces between L^p and $\mathrm{dom}(A_p(\mu)+1)$, see [89, Ch. 1.15.3] or [7, Ch. 4.6/4.7]. We devote a subchapter to the special fractional powers which we need in the following.

3.2. Fractional powers of the elliptic operators

In this section, we ultimately establish the embedding

$$\mathrm{dom}\big((A_p(\mu)+1)^{\frac12+\frac{d}{2q}}\big) \hookrightarrow W^{1,q} \tag{3.4}$$

for some $q > d$ with $p \geq \frac{q}{2}$, cf. Theorem 3.10 below. The main tool here, which will be the "anchor" in the derivation of (3.4), is the precise information on the domain of definition of the square root of the operators $-\nabla \cdot \mu\nabla + 1$, cf. Proposition 3.8, together with the following assumption, which essentially allows to "lift" the obtained regularity to sufficiently high levels:

Assumption 3.6. There is a $q > d$ such that

$$-\nabla \cdot \nabla + 1 \colon W^{1,q} \to W_{\bullet}^{-1,q} \tag{3.5}$$

provides a topological isomorphism, the operator being defined as in Definition 3.1. Equivalently, (3.5) being a continuous isomorphism means that $\mathrm{dom}_{W_{\bullet}^{-1,q}}(-\nabla\cdot\nabla+1)$ is exactly $W^{1,q}$.

We suppose Assumption 3.6 to be satisfied for the rest of this work and fix the corresponding number $q \in \,]d, 4[$.

Since Assumption 3.6 in fact implicitly determines the class of admissible domains, an (extensive) comment on this should be in order:

Remark 3.7. i) In case of $d = 2$, the assumption is fulfilled for any Lipschitz domain Ω. This is the main result in the classical paper [40], there even established for mixed boundary conditions.

ii) It is exactly this condition which—besides the *a priori* required Lipschitz property—puts a restriction on the geometry of the underlying domain Ω in three spatial dimensions in this paper. For $d = 3$, it is known that Assumption 3.6 holds true in case of *strong* Lipschitz domains Ω, cf. [98]. Moreover, it is also true for Lipschitz domains Ω whose closures form—generally nonconvex—polyhedrons, cf. [41]. Note that this latter class is, by far, *not* contained in the class of strong Lipschitz domains, as the (topologically regularized) double beam shows.

iii) Assumption 3.6 is also fulfilled for domains which are obtained locally as C^1 deformations of the ones mentioned before.

iv) It is well-known that, even for strong Lipschitz domains, the admissible index q exceeds 3 by an arbitrarily small margin only, cf. [98, Introduction], cf. also [57, Thm. A]. In case of C^1-domains Ω, q may be chosen arbitrarily large (cf. [1, Section 15] or [71, p. 156–157]); but if one admits polyhedral domains the isomorphism index q cannot be expected to be larger than 4 in general, since edge and corner singularities appear, cf. [23], [24]. See also [70] and [42, Appendix] for sharp estimates of edge singularities.

v) If ϕ is a uniformly continuous function on Ω with a positive lower bound, then Assumption 3.6 implies that

$$-\nabla \cdot \phi\nabla + 1 \colon W^{1,q} \to W_{\bullet}^{-1,q} \tag{3.6}$$

is also a topological isomorphism, cf. [28, Ch. 6].

Altogether, this shows that Assumption 3.6 is fulfilled for a fairly rich class of domains which should cover almost all interesting constellations in the applications.

The following recent result on the regularity properties of the square root of $-\nabla \cdot \mu\nabla + 1$ is, in cooperation with the isomorphism (3.5), the central instrument for deriving estimates for suitable fractional powers of the differential operators.

Proposition 3.8. *Let μ denote any real, measurable function on Ω which is bounded from below and above by positive constants.*

i) The isomorphism $(A_2(\mu)+1)^{-\frac{1}{2}}\colon L^2 \to W^{1,2}$*, cf. Proposition 3.4, continuously extends to an isomorphism from* L^p *onto* $W^{1,p}$ *for* $p \in \,]1,2[$*. Hence, the operator* $(A_p(\mu)+1)^{\frac{1}{2}}$ *provides a topological isomorphism between the spaces* $W^{1,p}$ *and* L^p*, or, in other words:* $\mathrm{dom}(A_p(\mu)+1)^{\frac{1}{2}} = W^{1,p}$*, for all* $p \in \,]1,2[$*.*

ii) $(-\nabla\cdot\mu\nabla+1)^{\frac{1}{2}}$ *provides a topological isomorphism between the spaces* L^p *and* $W^{-1,p}_\bullet$*, in other words:* $\mathrm{dom}_{W^{-1,p}_\bullet}(-\nabla\cdot\mu\nabla+1)^{\frac{1}{2}} = L^p$*, for all* $p \in [2,\infty[$*.*

iii) We have

$$\mathrm{dom}\big((A_p(\mu)+1)^{\frac{\theta}{2}}\big) = H^{\theta,p} \tag{3.7}$$

for $p \in \,]1,2]$ *and* $\theta \in \,]0,1[\,\setminus\{\frac{1}{p}\}$*.*

Proof. i) is the main result in [10], cf. Thm. 5.1 there. ii) follows from i) by duality because $A_2(\mu)$ is selfadjoint on L^2, see Proposition 3.4. iii) Since $A_p(\mu)+1$ admits bounded imaginary powers (again, Proposition 3.4),

$$\mathrm{dom}\big((A_p(\mu)+1)^{\frac{\theta}{2}}\big) = \big[L^p, \mathrm{dom}(A_p(\mu)+1)^{\frac{1}{2}}\big]_\theta$$

follows from [89, Ch. 1.15.3]. By i), the latter is equal to $[L^p, W^{1,p}]_\theta$, and this space is exactly $H^{\theta,p}$ as proved in [37, Thm. 3.1]. □

Lemma 3.9. *Let* ϕ *denote any real, uniformly continuous function on* Ω *which is bounded from below by a positive constant. Then, under Assumption 3.6,* $\big(A_p(\phi)+1\big)^{\frac{1}{2}}$ *provides a topological isomorphism between* $W^{1,p}$ *and* L^p *for all* $p \in \,]2,q]$*, and 3.7 for* $\mu = \phi$ *holds true for this range of* p *as well.*

Proof. First of all, Remark 3.7 tells us that under the given supposition on ϕ, Assumption 3.6 implies the isomorphism property (3.6), which then also holds true for all $p \in [2,q]$ due to interpolation. Having this at hand, the isomorphism property for the square root operators follows in a straight forward manner from Proposition 3.8 ii) for $\mu = \phi$, see also [27, Thm. 6.5]. This also implies (3.7) for $p \in \,]2,q]$ with the same proof as in Proposition 3.8. □

The square root isomorphisms and identity (3.7) from Lemma 3.9 have the following immediate consequence:

Theorem 3.10. *Let* ϕ *denote any real, uniformly continuous function on* Ω *which is bounded from below by a positive constant. Then, for every* $p \geq \frac{q}{2}$ *one has the embedding*

$$\mathrm{dom}\big((A_p(\phi)+1)^{\frac{1}{2}+\frac{d}{2q}}\big) \hookrightarrow W^{1,q}, \tag{3.8}$$

which implies

$$\big(L^p, \mathrm{dom}(A_p(\phi))\big)_{\theta,1} \hookrightarrow W^{1,q}, \tag{3.9}$$

for all $\theta \in \big[\frac{1}{2}+\frac{d}{2q}, 1\big[$*.*

Proof. The bounded imaginary powers of $A_p(\phi)+1$, cf. Proposition 3.4, imply that

$$\big(L^p, \mathrm{dom}(A_p(\phi)+1)\big)_{\theta,1} \hookrightarrow \big[L^p, \mathrm{dom}(A_p(\phi)+1)\big]_{\frac{1}{2}+\frac{d}{2q}} = \mathrm{dom}\big((A_p(\phi)+1)^{\frac{1}{2}+\frac{d}{2q}}\big)$$

for all $\theta \in \big[\frac{1}{2}+\frac{d}{2q}, 1\big[$, see [89, Ch. 1.15.3]. In this sense, (3.9) is a direct consequence of (3.8), modulo identification of $\mathrm{dom}\, A_p(\phi)$ and $\mathrm{dom}\big(A_p(\phi)+1\big)$. We show that (3.8) holds true by proving that $(A_p(\phi)+1)^{-(\frac{1}{2}+\frac{d}{2q})}$ is a continuous linear operator from L^p to $W^{1,q}$ for these θ. We split the operator as follows:

$$\begin{aligned}\big\|(A_p(\phi)+1)^{-(\frac{1}{2}+\frac{d}{2q})}\big\|_{\mathcal{L}(L^p;W^{1,q})}&\\ \leq \big\|(A_q(\phi)+1)^{-\frac{1}{2}}\big\|_{\mathcal{L}(L^q;W^{1,q})}\big\|(A_p(\phi)+1)^{-\frac{d}{2q}}\big\|_{\mathcal{L}(L^p;L^q)}.&\end{aligned} \tag{3.10}$$

Thanks to Lemma 3.9, it remains to show that $(A_p(\phi)+1)^{-\frac{d}{2q}}$ is a continuous linear operator from L^p to L^q. We show that $\operatorname{dom}\big((A_p(\phi)+1)^{\frac{d}{2q}}\big) \hookrightarrow L^q$. For $p>q$, we always have (cf. [89, Thm. 1.15.2])

$$\operatorname{dom}\big((A_p(\phi)+1)^{\theta}\big) \hookrightarrow \big(L^p, \operatorname{dom}(A_p(\phi)+1)\big)_{\theta,\infty} \hookrightarrow L^p \hookrightarrow L^q.$$

For $p \in \big[\frac{q}{2}, q\big]$ in turn, Proposition 3.8 and Lemma 3.9 yield $\operatorname{dom}((A_p(\phi)+1)^{\frac{d}{2q}}) = H^{\frac{d}{q},p}$ which exactly embeds into L^q. Hence, $(A_p(\phi)+1)^{-\frac{d}{2q}} \in \mathcal{L}(L^p;L^q)$ in both cases, and from (3.10) we obtain that

$$\operatorname{dom}\big((A_p(\phi)+1)^{\frac{1}{2}+\frac{d}{2q}}\big) \hookrightarrow W^{1,q},$$

which was the claim. □

3.3. Maximal parabolic regularity and consequences for nonlinear problems

We next introduce preparatory concepts and results concerning parabolic operators. Throughout the rest of this paper let $T>0$ and set $J = \,]0,T[$. First, we introduce the Bochner-Sobolev spaces.

Definition 3.11. If X is a Banach space and $r \in \,]1,\infty[$, then we denote by $L^r(J;X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J \|f(t)\|_X^r \,\mathrm{d}t$ is finite. We define the Bochner-Sobolev spaces

$$W^{1,r}(J;X) := \big\{u \in L^r(J;X)\colon u' \in L^r(J;X)\big\},$$

where u' is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [7, Section III.1]). Moreover, we introduce the subspace of functions with initial value zero $W_0^{1,r}(J;X) := \{\psi \in W^{1,r}(J;X)\colon \psi(0)=0\}$.

Let us define a suitable notion of maximal parabolic regularity in the non-autonomous case and point out some basic facts on this:

Definition 3.12. Let X, D be Banach spaces with D densely embedded in X. Let $J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D;X)$ be a bounded and measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in X for all $t \in J$. Let $r \in \,]1,\infty[$. Then we say that the family $\{\mathcal{A}(t)\}_{t\in J}$ satisfies *(non-autonomous) maximal parabolic $L^r(J;D,X)$-regularity*, if for any $f \in L^r(J;X)$ there is a unique function $u \in L^r(J;D) \cap W_0^{1,r}(J;X)$ which satisfies

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \tag{3.11}$$

for almost all $t \in J$. We write

$$\mathrm{MR}^r(J;D,X) := L^r(J;D) \cap W^{1,r}(J;X)$$

and

$$\mathrm{MR}_0^r(J;D,X) := L^r(J;D) \cap W_0^{1,r}(J;X)$$

for the spaces of maximal parabolic regularity. From the open mapping theorem, we further obtain that there exists a constant c such that

$$\|u\|_{\mathrm{MR}_0^r(J;D,X)} \le c\|f\|_{L^r(J;X)} \tag{3.12}$$

for all $f \in L^r(J;X)$ and u being the associated unique solution of (3.11).

If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator $\mathcal{A}_0$, and there exists an $r \in \,]1,\infty[$ such that $\{\mathcal{A}(t)\}_{t\in J}$ satisfies maximal parabolic $L^r(J;D,X)$-regularity, then $\{\mathcal{A}(t)\}_{t\in J}$ satisfies maximal parabolic $L^s(I;D,X)$-regularity for all $s \in \,]1,\infty[$ and all other (finite) intervals I (cf. [29]), and we say that $\mathcal{A}_0$ satisfies *maximal parabolic regularity on X*.

The following embedding result for the spaces of maximal parabolic regularity is essentially used in the sequel.

Lemma 3.13. *Let X, Y be two Banach spaces, with dense embedding $Y \hookrightarrow X$, and let $r \in \,]1,\infty[$.*

i) There is an embedding

$$\mathrm{MR}^r(J;Y,X) \hookrightarrow C\big(\overline{J};(X,Y)_{1-\frac{1}{r},r}\big). \tag{3.13}$$

ii) Conversely, if the operator A generates an analytic semigroup on the Banach space X with Y as its domain, and $\psi \in (X,Y)_{1-\frac{1}{s},s}$, then the function $\exp(\cdot A)\,\psi$ *belongs to $W^{1,s}(J;X) \cap L^s(J;Y)$ for every bounded interval interval $J = [0,T[$.*

iii) There is an embedding

$$\mathrm{MR}^r(J;Y,X) \hookrightarrow C^\alpha\big(J;(X,Y)_{\varrho,1}\big) \tag{3.14}$$

where $0 < \alpha = 1 - \varrho - \frac{1}{r}$.

Proof. i) is proved in [7, Ch. 4.10], ii) is shown in [66, Ch. 2.2.1 Prop. 2.2.2], and iii) is proved in [6, Ch. 3, Thm. 3], see also [26] for a simple proof. □

In the immediate context of maximal parabolic regularity, Y is taken as $\mathrm{dom}_X(A)$ equipped with the graph norm, of course.

Remark 3.14. The first two points of Lemma 3.13 together show that the space $(X, \mathrm{dom}_X(A))_{1-\frac{1}{r},r}$, is the adequate space of initial values in the framework of maximal parabolic regularity.

Moreover, we need the following results.

Theorem 3.15 ([84, Thm. 2.5])**.** *Let the following two suppositions be satisfied:*

(H1) The family of operators $\{\mathcal{A}(t)\}_{t\in\overline{J}}$, acting on a Banach space X has a common dense domain D and the mapping $\overline{J} \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D;X)$ is continuous. Moreover, each operator $\mathcal{A}(\tau)$, $\tau \in \overline{J}$, generates an analytic semigroup on X.

(H2) For some $r \in \,]1,\infty[$, every (fixed) $\tau \in [0,T]$ and all $f \in L^r(J;X)$ there is a unique element $u \in \mathrm{MR}_0^r(J;D;X)$ which satisfies the equation $u' + \mathcal{A}(\tau)u = f$.

Then $\{\mathcal{A}(t)\}_{t\in\overline{J}}$ satisfies maximal parabolic $L^r(J;D,X)$-regularity.

Theorem 3.16. *Let μ be a real, bounded, measurable function on Ω which admits a positive lower bound. Then, for every $p \in]1, \infty[$, the operators $A_p(\mu)$ admit maximal parabolic regularity on L^p.*

Proof. The theorem can be proved in different ways: in [45, Thm. 5.4] it is shown via Gaussian estimates for the heat kernel, heavily resting on [44], see also [21]. Alternatively, the theorem is proved in [38, Ch. 7], there resting on the contractivity of the induced semigroups on all L^p spaces (cf. Proposition 3.4) and the pioneering result of Lamberton [64]. The latter allows to prove maximal parabolic regularity on even more general Lebesgue spaces, see [32]. □

Theorem 3.17 ([4, Thm. 2.1])**.** *Let $r \in]1, \infty[$ and suppose that X, Y are Banach spaces with dense embedding $Y \hookrightarrow X$. Also assume the following:*

i) *$\mathcal{A}$ is a map from $\mathrm{MR}^r(J; Y, X)$ into $L^\infty(J; \mathcal{L}(Y; X))$, the latter space being identified with a subset of the* non-autonomous *parabolic operators on X. Moreover, $\mathcal{A}$ is Lipschitz continuous on bounded subsets.*

ii) *For each $u \in \mathrm{MR}^r(J; Y, X)$ and every $S \in]0, T]$ the non-autonomous operator $\mathcal{A}(u)|_{]0,S[}$ provides a topological isomorphism between $\mathrm{MR}^r_0(0, S; Y, X)$ and $L^r(0, S; X)$.*

iii) *The mapping $F \colon \mathrm{MR}^r(J; Y, X) \to L^s(J; X)$ is Lipschitzian on every bounded subset for some $s > r$.*

iv) *Both $\mathrm{MR}^r(J; Y, X) \ni u \mapsto \mathcal{A}(u) \in L^\infty(J; \mathcal{L}(Y; X))$ and $F \colon \mathrm{MR}^r(J; Y, X) \to L^s(J; X)$ are* Volterra maps*, i.e.*

$$u|_{]0,S[} = v|_{]0,S[} \quad \Longrightarrow \quad \big(A(u), F(u)\big)|_{]0,S[} = \big(A(v), F(v)\big)|_{]0,S[}$$

for every $S \in]0, T[$.

v) *$u_0 \in (X, Y)_{1-\frac{1}{r}, r}$.*

Then there is a (maximal) interval $I_\bullet :=]0, S_\bullet[\subseteq J$ such that the equation

$$u' + \mathcal{A}(u)u = F(u), \quad u(0) = u_0$$

has a solution u on every subinterval $I =]0, S[\subseteq I_\bullet$ which belongs to the maximum regularity space $\mathrm{MR}^r(I; Y, X)$. Moreover, this solution is unique.

Remark 3.18. It is known since long that the Volterra property allows to derive results which are not available in a more general context without this property, see e.g. [33, Ch. V]. Nevertheless, we feel that Amann's result is very close to the "optimum" what can be achieved. The reader is advised to consult [5, Thm. 3.1] for comments on the result by its inventor and a (fixable) shortcoming in the proof in [4].

3.4. Transferring to real spaces

Up to now, we have worked in a *complex* setting, but the Keller-Segel system has to be read as a *real* one. Therefore we transfer the results which we need in the sequel to the corresponding real spaces. In order to do this, we denote the real parts of L^p and $W^{1,q}$ by $L^p_{\mathbb{R}}$ and $W^{1,q}_{\mathbb{R}}$.

Remark 3.19. The necessity to start with complex spaces and to re-evaluate the assertions to also hold in the real case can be explained as follows: Most results up to this chapter 3.4 are complex in their very nature, a particular example being Proposition 3.8. This makes it evident that, at this point, complex spaces are the correct setting. On the other hand, the condition of being twice continuously differentiable for the nonlinear functions is more or less inevitable in our context as will become clear below, cf. Lemma 4.14, Corollary 4.15 and Lemma 4.16. But imposing this condition in a *complex* setting in fact necessitates the *analyticity* of the corresponding functions, which is drastically and more importantly unnecessarily more restrictive. Hence we "do the twist" and switch to real spaces for the actual investigation of the model.

The starting point is the insight that the semigroup operators $\exp(-tA_p(\mu))$ map real functions into real functions if the coefficient function μ is real-valued, as noted in Proposition 3.4. Hence, the operators $(A_p(\mu)+\lambda)^{-1}\colon L^p \to L^p$ *also* map real functions into real ones if $\lambda \in]0,\infty[$. This makes clear that the operator $A_p(\mu)$ has a meaningful restriction to $L^p_{\mathbb{R}}$, whose domain also consists of real functions only. We denote this domain by $\mathrm{dom}_{\mathbb{R}}(A_p(\mu))$ for the rest of this subsection.

Lemma 3.20. *Let ϕ be a real, uniformly continuous function which is bounded from below by a positive constant. The assertion of Theorem 3.10 remains true in case of real spaces, i.e., one has for $p \geq \frac{q}{2}$ the embedding*

$$\big(L^p_{\mathbb{R}}, \mathrm{dom}_{\mathbb{R}}(A_p(\phi))\big)_{\theta,1} \hookrightarrow W^{1,q}_{\mathbb{R}} \hookrightarrow C(\overline{\Omega}) \tag{3.15}$$

for all $\theta \in \big[\frac{1}{2} + \frac{d}{2q}, 1\big[$.

Proof. Let us first recall (see Remark 3.5) that we have topologized $\mathrm{dom}_{\mathbb{R}}(A_p(\phi))$ by the norm $\|(A_p(\phi)+1)\cdot\|_{L^p_{\mathbb{R}}}$. Further, by Theorem 3.10, there is a positive constant c such that the following inequality holds true for all $\psi \in \mathrm{dom}(A_p(\phi))$ and $\theta \in [\frac{1}{2}+\frac{d}{2q}, 1[$:

$$\|\psi\|_{W^{1,q}} \leq c\,\|\psi\|_{L^p}^{1-\theta}\,\|\psi\|^{\theta}_{\mathrm{dom}(A_p(\phi))} = c\,\|\psi\|_{L^p}^{1-\theta}\,\big\|(A_p(\phi)+1)\psi\big\|^{\theta}_{L^p}. \tag{3.16}$$

In particular, inequality (3.16) is true for every *real* function $\psi \in \mathrm{dom}_{\mathbb{R}}(A_p(\phi))$, and then reads

$$\|\psi\|_{W^{1,q}_{\mathbb{R}}} \leq c\,\|\psi\|_{L^p_{\mathbb{R}}}^{1-\theta}\,\big\|(A_p(\phi)+1)\psi\big\|^{\theta}_{L^p_{\mathbb{R}}} = c\,\|\psi\|_{L^p_{\mathbb{R}}}^{1-\theta}\|\psi\|^{\theta}_{\mathrm{dom}_{\mathbb{R}}(A_p(\phi))}. \tag{3.17}$$

But (3.17) is constitutive for the embedding (3.15), cf. [12, Ch. 3.5] or [11, Ch. 5, Prop. 2.10]. ☐

Theorem 3.21. *Let μ be a real, bounded, measurable function on Ω which admits a positive lower bound. Then, for every $p \in]1,\infty[$, $A_p(\mu)$ admits maximal parabolic $L^p_{\mathbb{R}}$ regularity.*

Proof. Let $f \in L^p_{\mathbb{R}}$. Then, by maximal parabolic L^p regularity of $A_p(\mu)$, there exists a unique function $u \in \mathrm{MR}^r_0(J;\mathrm{dom}(A_p(\mu)), L^p)$ such that

$$u'(t) + A_p(\mu)u(t) = f(t) \quad \text{in } L^p \quad \text{for almost all } t \in J.$$

But then this solution is given by the variation-of-constants formula

$$u(t) = \int_0^t \exp\big(-(t-s)A_p(\mu)\big)\, f(s)\,\mathrm{d}s,$$

and since the semigroup operators transform real functions into real ones, cf. Proposition 3.4, it is clear that the solution in fact belongs to the space $W_0^{1,r}(J; L^p_{\mathbb{R}}) \cap L^r(J; \mathrm{dom}_{\mathbb{R}}(A_p(\mu)))$, what proves the claim. $\square$

Switching to real spaces, the symbol $\mathrm{dom}(A_p(\mu))$ from now on denotes the domain of $A_p(\mu)$ considered on the *real* space $L^p_{\mathbb{R}}$.

3.5. Constant domains for $A_p(\varphi)$

We will need that the domains of the differential operators $A_p(\varphi)$ are uniform w.r.t. φ from a certain regularity class, as per the assumptions in Theorem 3.17. In general, this is not to be expected if φ does not have a positive lower bound, cf. also Remark 3.2. Still, we need that the differential operator on the right-hand side in (1.1), which is the one having potentially nonpositive coefficient function values, is compatible with the domain of definition for the function $v(t)$.

It will turn out that both the latter *and* the constant domain of definition for the differential operators on the left-hand side in (1.1) is exactly $\mathrm{dom}_{L^p}(\Delta)$. We prove the following lemma which covers all these considerations in its generality, there writing Δ instead of $-A_p(1)$ and already supposing that all occurring spaces are in fact *real* ones.

Lemma 3.22. *Let $p = \frac{q}{2}$ and assume $\rho \in W^{1,q}$. Then the following assertions hold true:*

i) The domain of the Laplacian is embedded into the domain of $A_p(\rho)$, that is,

$$\mathrm{dom}_{L^p}(\Delta) \hookrightarrow \mathrm{dom}(A_p(\rho)).$$

ii) If ρ has, additionally, a positive lower bound, then the reverse embedding

$$\mathrm{dom}(A_p(\rho)) \hookrightarrow \mathrm{dom}_{L^p}(\Delta)$$

is also true, and $\mathrm{dom}_{L^p}(\Delta)$ and $\mathrm{dom}(A_p(\rho))$ coincide as Banach spaces.

Proof. i) Let $\psi \in \mathrm{dom}_{L^p}(\Delta)$ and consider the linear form

$$\big(W^{1,2} \cap L^{p'}\big) \ni \varphi \mapsto \langle -\nabla \cdot \rho \nabla \psi, \varphi \rangle. \tag{3.18}$$

We show that $\psi \in \mathrm{dom}(A_p(\rho))$ by showing that (3.18) is continuous w.r.t. the $L^{p'}$-topology. Therefore we estimate

$$\begin{aligned} \Big| \int_\Omega \rho \nabla \psi \cdot \nabla \varphi \,\mathrm{dx} \Big| &= \Big| \int_\Omega \nabla \psi \cdot \nabla(\rho\varphi) \,\mathrm{dx} - \int_\Omega \varphi \nabla \psi \cdot \nabla \rho \,\mathrm{dx} \Big| \\ &\le \Big| \int_\Omega \nabla \psi \cdot \nabla(\rho\varphi) \,\mathrm{dx} \Big| + \Big| \int_\Omega \varphi \nabla \psi \cdot \nabla \rho \,\mathrm{dx} \Big| \\ &\le \|\rho\|_{L^\infty} \|\Delta\psi\|_{L^p} \|\varphi\|_{L^{p'}} + \|\nabla\psi\|_{L^q} \|\nabla\rho\|_{L^q} \|\varphi\|_{L^{p'}}. \end{aligned} \tag{3.19}$$

Since $\mathrm{dom}_{L^p}(\Delta)$ was topologized by $\|(-\Delta+1)\cdot\|_{L^p}$, we thus find

$$\begin{aligned} &\sup_{\varphi \in W^{1,2} \cap L^{p'}, \|\varphi\|_{L^{p'}} \le 1} \Big| \int_\Omega \rho \nabla \psi \cdot \nabla \varphi \,\mathrm{dx} \Big| \\ &\le \Big(\|\rho\|_{L^\infty} \big\|\Delta(-\Delta+1)^{-1}\big\|_{\mathcal{L}(L^p)} + \mathcal{E}\big(\mathrm{dom}_{L^p}(\Delta), W^{1,q}\big) \|\nabla\rho\|_{L^q} \Big) \|\psi\|_{\mathrm{dom}_{L^p}(\Delta)} \end{aligned} \tag{3.20}$$

This means that the linear form (3.18) is bounded on $(W^{1,2}, \|\cdot\|_{L^{p'}})$, such that $\psi \in \operatorname{dom}(A_p(\rho))$ by the construction in Chapter 3. Moreover, $\|A_p(\rho)\psi\|_{L^p}$ is bounded by the right-hand side in (3.20). The embedding $\operatorname{dom}_{L^p}(\Delta) \hookrightarrow \operatorname{dom}(A_p(\rho))$ follows immediately.

ii) One reasons analogously as in the previous case, but exploits instead of (3.19) the equality

$$\int_\Omega \nabla\psi \cdot \nabla\varphi \,\mathrm{dx} = \int_\Omega \rho^{-1}\rho\nabla\psi \cdot \nabla\varphi \,\mathrm{dx} = \int_\Omega \rho\nabla\psi \cdot \nabla(\rho^{-1}\varphi)\,\mathrm{dx} - \int_\Omega \varphi\rho\nabla\psi\nabla(\rho^{-1})\,\mathrm{dx}.$$

This gives $\operatorname{dom}(A_p(\rho)) \hookrightarrow \operatorname{dom}_{L^p}(\Delta)$, from which the Banach space identity $\operatorname{dom}_{L^p}(\Delta) = \operatorname{dom}(A_p(\rho))$ follows. □

Corollary 3.23. *For $p = \frac{q}{2}$, the mapping*

$$C\big(\overline{J}; W^{1,q}\big) \ni \omega \mapsto -\nabla\cdot\omega(\cdot)\nabla$$

takes its values in the space $C\big(\overline{J}; \mathcal{L}(\operatorname{dom}_{L^p}(\Delta); L^p)\big)$ and is Lipschitzian on bounded subsets.

4. Investigation of the model

4.1. Precise formulation of the problem and main result

In this section, we give a rigorous analysis of (1.1)–(1.6) in the sense of Definition 4.1 below. In fact, most of this section will consist of the proof of the main Theorem 4.3, which we state in the following. An explanation of the strategy for the proof can be found in Section 4.2.

Let us first agree on the following: All appearing function spaces are supposed to be *real* ones, without indicating this explicitly in the sequel.

For all what follows, we suppose Assumption 2.1 to be satisfied. We moreover fix $p = \frac{q}{2}$ with q being the number from Assumption 3.6, which is also assumed to hold true. We abbreviate $A_p(\mu)$ *for this fixed* p by $A(\mu)$ for a measurable, bounded and real coefficient function μ. Fix also a number $r > 2(1-\frac{d}{q})^{-1}$ and $s > r$.

In the following we want to establish a precise notion of a *solution of the Keller-Segel-Model*.

Definition 4.1. Given a subinterval $I = \,]0, S[$ of J, we call a quadruple of functions

$$\big(u, (v, p, w)\big) \in \mathrm{MR}^r(I; \operatorname{dom}_{L^p}(\Delta), L^p) \times \mathrm{MR}^s(I; \operatorname{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$$

a general solution of (1.1)–(1.6) on I, if these satisfy

$$\begin{aligned} u'(t) + A\big(\kappa(u(t), v(t))\big)u(t) &= A\big(\sigma(u(t), v(t))\big)v(t) \\ &\quad + R_1\big(u(t), v(t), p(t), w(t)\big) && (4.1) \\ v'(t) - k_v\Delta v(t) &= R_2\big(u(t), v(t), p(t), w(t)\big) && (4.2) \\ p'(t) - k_p\Delta p(t) &= R_3\big(u(t), v(t), p(t), w(t)\big) && (4.3) \\ w'(t) - k_w\Delta w(t) &= R_4\big(u(t), v(t), p(t), w(t)\big) && (4.4) \\ \big(u(0), v(0), p(0), w(0)\big) &= (u_0, v_0, p_0, w_0) && (4.5) \end{aligned}$$

for almost all $t \in I$ in $L^p \times \mathbb{L}^p$ for (4.1)–(4.4), where the time derivative is taken in the sense of vector valued distributions and the initial values satisfy

$$(u_0, v_0, p_0, w_0) \in \big(L^p, \mathrm{dom}_{L^p}(\Delta)\big)_{1-\frac{1}{r},r} \times \Big(\big(L^p, \mathrm{dom}_{L^p}(\Delta)\big)_{1-\frac{1}{s},s}\Big)^3 =: \mathrm{IV}(r,s).$$

The operator $-\Delta$ is here to be understood as $A_p(1)$, i.e., the restriction of the weak (negative) Laplacian to L^p.

Remark 4.2. i) In the original model, we had the specific inhomogeneities

$$\begin{aligned} R_1(u,v,p,w) &= 0, \\ R_2(u,v,p,w) &= -r_1 vp + r_{-1} w + u f(v), \\ R_3(u,v,p,w) &= -r_1 vp + (r_{-1} + r_2) w + u g(v,p), \\ R_4(u,v,p,w) &= r_1 vp - (r_{-1} + r_2) w, \end{aligned}$$

cf. (1.1)–(1.4). If f and g are continuously differentiable as real functions, this choice clearly satisfies the assumptions on the functions R_i as in Assumption 2.1.

ii) For almost all $t \in I$ the functions $u(t,\cdot), v(t,\cdot), p(t,\cdot), w(t,\cdot)$ each lie in the space $\mathrm{dom}_{L^p}(\Delta)$, hence for these t a *homogeneous Neumann condition* is fulfilled in a generalized sense, cf. Remark 3.3.

iii) The regularity of the initial values in $\mathrm{IV}(r,s)$ is exactly the optimal one for the class of solutions as defined in Definition 4.1, cf. Remark 3.14.

iv) Definition 4.1 is in fact faithful to itself in the sense that the functions and mappings indeed map into the correct spaces, see also Remark 4.6 below.

We formulate now the main result of this work.

Theorem 4.3. *Under Assumption 3.6, problem* (1.1)–(1.6) *admits exactly one local-in-time general solution in the spirit of Definition 4.1. Moreover, the solutions* (v,p,w) *are uniformly bounded in* L^∞ *over the maximal interval of existence.*

Remark 4.4. Considering the derivation of the model in the introductory chapter, the question of *positivity* of the solutions (u,v,p,w) in the sense of Definition 4.1—provided their initial values were positive in the first place—arises naturally. It is a standard result in the theory of reaction-diffusion systems (cf. e.g. [82]) that a system in the form (4.2)–(4.4) is positivity preserving if and only if the inhomogeneities $R_2(\bar u,\cdot), R_3(\bar u,\cdot), R_4(\bar u,\cdot)$ are *quasipositive* for every $\bar u \in \mathbb{R}$, that is, if $(\bar v, \bar p, \bar w)$ is an arbitrary vector in $\mathbb{R}^3$ with nonnegative entries, then

$$R_2(\bar u, 0, \bar p, \bar w) \geq 0, \quad R_3(\bar u, \bar v, 0, \bar w) \geq 0 \quad \text{and} \quad R_4(\bar u, \bar v, \bar p, 0) \geq 0.$$

The specific inhomogeneities in (1.2)–(1.4), cf. Remark 4.2, indeed satisfy this condition if $g(\bar v, 0) \geq 0$ and $f(\bar u) \geq 0$ for nonnegative $\bar v, \bar u \geq 0$. Hence, (4.2)–(4.4) is positivity preserving for (v,p,w) if u is also a positive function, i.e., (4.1) is *also* positivity preserving. Unfortunately, the latter seems very difficult to show in the very general context of Definition 4.1, even with $R_1 = 0$, and is generally not true for seemingly easy cases, see [75, Ch. 5]. However, for the specific choices $\kappa(u,v) = 1$ and $\sigma(u,v) = -u$, already mentioned in the introduction as well-researched model choices, positivity of u is shown in [34, Thm. 3.3] *independent* of the sign of v. The proof in [34] only relies on the fact that v is uniformly bounded in time and space, which is the case for our solutions obtained from Theorem 4.3. Hence, for this choice of κ and

σ, $R_1 = 0$ and $R_2(\bar{u},\cdot), R_3(\bar{u},\cdot), R_4(\bar{u},\cdot)$ quasipositive for $\bar{u} \geq 0$, system (4.1)–(4.4) is indeed positivity preserving. This includes in particular system (1.1)–(1.4) for this choice of κ and σ and f, g as mentioned above.

We now proceed with the proof of the main result.

4.2. The proof

The actual proof of Theorem 4.3 works in as follows. It should be evident to the reader that we plan to use the abstract result of Amann, Theorem 3.17. The general idea is to solve the semilinear equations for (v,p,w), (4.2)–(4.4), in dependence of u, and to show that this dependence re-inserted in the first equation for u satisfies the assumptions in Theorem 3.17. Here, it is clear that the dependence of (v,p,w) on u will be *nonlocal in time*, which indeed makes Theorem 3.17—instead of other well-known abstract quasilinear existence results—necessary.

However, as (4.2)–(4.4) are *nonlinear* equations themselves, it is not *a priori* clear that they in fact admit global solutions on the whole time horizon $]0,T[$, and a local-in-time existence interval $I(u)$ for (v,p,w) depending on u would clearly thwart any attempt to establish the assumptions from Theorem 3.17. Hence, we modify the right-hand sides in (4.2)–(4.4) by introducing a suitable cut-off, which then allows to show global existence, uniqueness, and a well-behaved dependence on u for the solutions $(\hat{v},\hat{p},\hat{w})$ of the modified lower system ((4.10)–(4.12) below); this is Theorem 4.10.

After establishing that the involved operators and functions satisfy the assumptions of Theorem 3.17, we then use that very theorem to show existence and uniqueness of a local-in-time solution u to the *modified* system, including the equation for u, in Theorems 4.13 and 4.9. From there, we finally obtain Theorem 4.3 by showing that the local-in-time solution obtained for the modified system is indeed also the solution to the original system (4.1)–(4.5) at the cost of a possibly still smaller existence interval.

Aside from the dependence of (v,p,w) on u, there is another major obstacle when working to satisfy the assumptions of Theorem 3.17: Assumption i) of said theorem in fact requires, in our notation, that the differential operators, which will be $A\bigl(\kappa(u(t),v(u)(t))\bigr)$, have *uniform* domains Y for *all* $u \in \mathrm{MR}^r(J;Y,L^p)$ and for almost every $t \in J$. Thanks to Lemma 3.22, we will be able to use $Y = \mathrm{dom}_{L^p}(\Delta)$, provided that the coefficient functions $\kappa(u(t),v(u)(t))$ are from $W^{1,q}$ for almost every $t \in J$. We have already laid the foundations to show this in Lemma 3.20, together with the maximal regularity embedding (3.13), which together immediately yield the following introductory result which is of importance in all what follows.

Lemma 4.5. *Set* $\alpha = \frac{1}{2} - \frac{d}{2q} - \frac{1}{r}$*. By the choice of* r*, we have* $\alpha > 0$*.*

i) The space $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ *embeds into* $C^\alpha(J;W^{1,q})$ *and, hence, compactly into* $C(\overline{J};C(\overline{\Omega}))$*.*

ii) Analogously, $\mathrm{MR}^s_0(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ *embeds into* $C^\alpha(J;\mathbb{W}^{1,q})$ *and, hence, compactly into* $C(\overline{J};C(\overline{\Omega})^3)$*.*

Proof. The compactness in both cases follows by the vector-valued Arzelà-Ascoli theorem, cf. [65, Ch. III.3]. For i), the condition on r implies $1 - \frac{1}{r} - \bigl(\frac{1}{2} + \frac{d}{2q}\bigr) > 0$. Thus, the claim follows from Lemma 3.13, cf. (3.14), in conjunction with Lemma 3.20. ii) is proved analogously. □

Remark 4.6. For $u \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ and $v \in \mathrm{MR}^s(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ with I as in Definition 4.1, Lemma 4.5 together with Lemma 3.22 and the assumptions on κ and σ (cf. Assumption 3.6) tells us that $\kappa(u(t), v(t))$ and $\sigma(u(t), v(t))$ are each functions from $W^{1,q}$ for every $t \in \overline{I}$. Together with $u(t), v(t) \in \mathrm{dom}_{L^p}(\Delta)$ for almost every $t \in I$, this shows that the expressions $A(\kappa(u(t), v(t)))u(t)$ and $A(\sigma(u(t), v(t)))v(t)$ in (4.1) are indeed well-defined. See also Lemmata 4.16 and 4.17 below.

We will now modify the abstract system (4.1)–(4.4) in such a way that the terms on the right hand sides of (4.2)–(4.4) become bounded in space and time. This will ultimately lead to a solution in the spirit of Definition 4.1 on a *smaller* time interval, since the modification becomes "active", only after some time point $T_\bullet > 0$, allowing to re-obtain the correct solution to the unmodified system on $[0, T_\bullet]$.

We consider

$$(v_0, p_0, w_0) \in \big((L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}\big)^3 \tag{4.6}$$

to be given and fixed from now on.

Definition 4.7. For $\delta > 0$, we put $M := \delta + \max(\|v_0\|_{L^\infty}, \|p_0\|_{L^\infty}, \|w_0\|_{L^\infty})$. Let $\eta \in C^\infty(\mathbb{R})$ be a smooth function which is the identity on the interval $[-M, M]$ and is equal to $-(M+1)$ on the interval $]-\infty, -(M+1)]$ and equal to $M+1$ on the interval $[M+1, \infty[$. Moreover, we put $R_i^\eta := R_i(\cdot, \eta(\cdot), \eta(\cdot), \eta(\cdot))$ for $i = 2, 3, 4$.

Note that, due to Lemma 3.20 and the choice of s, we have the embedding $(L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s} \hookrightarrow C(\overline{\Omega})$, such that the number M in Definition 4.7 is well-defined.

We further split off the initial values for the functions v, p, w for which we put $v_{\mathcal{I}}(t) = \exp(t\, k_v \Delta)\, v_0$ as well as $p_{\mathcal{I}}(t) = \exp(t\, k_p \Delta)\, p_0$ and $w_{\mathcal{I}}(t) = \exp(t\, k_w \Delta)\, w_0$, and write

$$v = v_{\mathcal{I}} + \check{v}, \quad p = p_{\mathcal{I}} + \check{p}, \quad w = w_{\mathcal{I}} + \check{w}, \tag{4.7}$$

where $\check{v}, \check{p}$ and $\check{w}$ have the initial value 0, of course.

For convenience, we collect some of the properties for the functions $v_{\mathcal{I}}, p_{\mathcal{I}}$ and $w_{\mathcal{I}}$ which will be of importance later.

Lemma 4.8. *Let the initial values (v_0, p_0, w_0) satisfy (4.6).*

i) One has

$$v_{\mathcal{I}}' - k_v \Delta v_{\mathcal{I}} = p_{\mathcal{I}}' - k_p \Delta p_{\mathcal{I}} = w_{\mathcal{I}}' - k_w \Delta w_{\mathcal{I}} \equiv 0 \tag{4.8}$$

on any time interval $]0, S[\subseteq J$.

ii) The functions $v_{\mathcal{I}}, p_{\mathcal{I}}$ and $w_{\mathcal{I}}$ are each from $\mathrm{MR}^s(J; \mathrm{dom}_{L^p}(\Delta), L^p)$, take their values pointwise on J in $W^{1,q}$, and are continuous on every time interval $[0, S[\subset \overline{J}$.

iii) The functions $v_{\mathcal{I}}, p_{\mathcal{I}}$ and $w_{\mathcal{I}}$ are continuous on every time interval $[0, S[\subset \overline{J}$ when considered as $C(\overline{\Omega})$-valued. Moreover, in this case we have

$$\|v_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \le \|v_{\mathcal{I}}(0)\|_{C(\overline{\Omega})}, \quad \|p_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \le \|p_{\mathcal{I}}(0)\|_{C(\overline{\Omega})},$$
$$\text{and} \quad \|w_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \le \|w_{\mathcal{I}}(0)\|_{C(\overline{\Omega})}$$

for every $s \in S$

Proof. i) is clear. ii) Lemma 3.13 ii) shows that the functions $v_{\mathcal{I}}, p_{\mathcal{I}}, w_{\mathcal{I}}$ are continuous when considered as $(L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}$-valued ones. Thus, the assertion follows from Lemma 3.20 and the definition of s. iii) The first assertion follows from ii) by embedding $W^{1,q} \hookrightarrow C(\overline{\Omega})$. Moreover, since the semigroups act as *contractive* ones in L^∞, cf. Proposition 3.4, the evolution of the initial values v_0, p_0, w_0 does not lead to larger L^∞-norms. The latter is identical with the $C(\overline{\Omega})$-norm in our case. □

Having introduced the modified nonlinearities R_i^η and the split-off of the initial values, we combine both into the functions $\widehat{R}_i \colon J \times C(\overline{\Omega}) \times \mathbb{L}^p \to L^p$ by

$$\widehat{R}_i(t; \mathfrak{u}, \mathfrak{v}, \mathfrak{p}, \mathfrak{w}) := R_i^\eta\big(\mathfrak{u}, v_{\mathcal{I}}(t) + \mathfrak{v}, p_{\mathcal{I}}(t) + \mathfrak{p}, w_{\mathcal{I}}(t) + \mathfrak{w}\big)$$

for $i = 2, 3, 4$, and

$$\widehat{R}_1(t; \mathfrak{u}, \mathfrak{v}, \mathfrak{p}, \mathfrak{w}) := R_1\big(\mathfrak{u}, v_{\mathcal{I}}(t) + \mathfrak{v}, p_{\mathcal{I}}(t) + \mathfrak{p}, w_{\mathcal{I}}(t) + \mathfrak{w}\big).$$

Then we consider instead of (4.1)–(4.5) the system

$$\begin{aligned} u'(t) + A\big(\kappa(u(t), v_{\mathcal{I}}(t) + v(t))\big)u(t) = A\big(\sigma(u(t), v_{\mathcal{I}}(t) + v(t)\big)\big(v_{\mathcal{I}}(t) + v(t)\big) \\ + \widehat{R}_1\big(t; u(t), v(t), p(t), w(t)\big), \qquad (4.9) \end{aligned}$$

$$v'(t) - k_v \Delta v(t) = \widehat{R}_2\big(t; u(t), v(t), p(t), w(t)\big), \qquad (4.10)$$

$$p'(t) - k_p \Delta p(t) = \widehat{R}_3\big(t; u(t), v(t), p(t), w(t)\big), \qquad (4.11)$$

$$w'(t) - k_w \Delta w(t) = \widehat{R}_4\big(t; u(t), v(t), p(t), w(t)\big), \qquad (4.12)$$

$$(u(0), v(0), p(0), w(0)) = (u_0, 0, 0, 0) \qquad (4.13)$$

as equations in the Banach space $L^p \times \mathbb{L}^p \times \mathrm{IV}(r,s)$, holding for almost every $t \in I$ for the first four components. Note that we have, by abuse of notation, returned to writing v, p and w instead of $\check{v}, \check{p}$ and $\check{w}$ as introduced in (4.7) for better readability. Since we work exclusively with the functions with initial value 0 from here on, this should not give rise to confusion to the reader.

After these preparations we prove the subsequent theorem, from which our main result, Theorem 4.3, then follows (and which is in fact only a slight reformulation of this).

Theorem 4.9. *For given* $(u_0, v_0, p_0, w_0) \in \mathrm{IV}(r,s)$*, the system* (4.9)–(4.13) *admits exactly one local-in-time solution*

$$\big(u, (v, p, w)\big) \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p) \times \mathrm{MR}^s_0(I; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p),$$

with $I = \,]0, S[\, \subseteq J$.

Let us re-iterate the strategy for the proof of Theorem 4.9: Firstly, we will solve the equations (4.10)–(4.12) with $u \in C(\overline{J}; C(\overline{\Omega}))$ fixed by a fixed-point argument. The crucial point is that the dependence of these solution (v, p, w) from u is well-behaved in the space $\mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$. So implicitly inserting this into (4.9), this equation decouples from the other ones and is tractable by means of Amann's result, Theorem 3.17. Having then u at hand (we prove that the assumptions of Theorem 3.17 are satisfied in Theorem 4.13), one "re-discovers" (v, p, w) by (4.10)–(4.12).

Theorem 4.10. *i) Assume* $u \in C(\overline{J}; C(\overline{\Omega}))$ *to be given. Then the system* (4.10)–(4.12) *has a unique solution* $(v,p,w) \in \mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$.

ii) Let $\mathcal{S}\colon C(\overline{J}; C(\overline{\Omega})) \to \mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ *denote the mapping which assigns to u the solution of* (4.10)–(4.12)*. Then* $\mathcal{S}$ *is continuously differentiable.*

Proof. i): For given $u \in C(\overline{J}; C(\overline{\Omega}))$, define $R_{i,u}\colon J \times \mathbb{L}^p \to L^p$, $i = 2,3,4$ by setting $R_{i,u}(t; v,p,w) := \widehat{R}_i(t; u(t), v,p,w)$. It is not hard to see that each $R_{i,u}$ is uniformly continuous on J when the second argument $(v,p,w) \in \mathbb{L}^p$ is fixed, and globally Lipschitz continuous on $\mathbb{L}^p$ when $t \in J$ is fixed – with a Lipschitz constant uniform in t. Therefore, the *semilinear* parabolic system (4.10)–(4.12) admits exactly one *mild* solution $(\hat v, \hat p, \hat w) \in C(\overline{J}; \mathbb{L}^p)$ with initial value zero, cf. [81, Ch. 6, Thm. 1.2]. Since then the mapping

$$J \ni t \mapsto \Big(R_{2,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big), R_{3,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big), R_{4,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big)\Big)$$

belongs to $L^\infty(J; \mathbb{L}^p)$, maximal parabolic regularity of the operator

$$-\widetilde{\Delta} := \mathrm{diag}(-k_v\Delta, -k_p\Delta, -k_p\Delta)$$

on $\mathbb{L}^p$ provides an unique solution $(\check v, \check p, \check w)$ with zero initial values of the equations

$$\begin{aligned} v'(t) - k_v\Delta v(t) &= R_{2,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big),\\ p'(t) - k_p\Delta p(t) &= R_{3,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big),\\ w'(t) - k_w\Delta w(t) &= R_{4,u}\big(t; \hat v(t), \hat p(t), \hat w(t)\big), \end{aligned}$$

which even belongs to the space $\mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$. But this solution $(\check v, \check p, \check w)$ is also a mild solution of (4.10)–(4.12), cf. [7, Ch. III.1.3]. Since then both $(\hat v, \hat p, \hat w)$ and $(\check v, \check p, \check w)$ are mild solutions of (4.10)–(4.12) with the same initial value, they must necessarily coincide. Hence, $(\hat v, \hat p, \hat w)$ belongs to $\mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ and is the unique function to solve (4.10)–(4.12).
ii) For this we apply the implicit function theorem, considering the mapping

$$\Psi\colon C(\overline{J}; C(\overline{\Omega})) \times \mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p) \to L^s(J; \mathbb{L}^p),$$

which is given by

$$\begin{aligned} \Psi(u,v,p,w)(t) = \Big(&v'(t) - k_v\Delta v(t) - R_{2,u}\big(t; v(t), p(t), w(t)\big),\\ &p'(t) - k_p\Delta p(t) - R_{3,u}\big(t; v(t), p(t), w(t)\big),\\ &w'(t) - k_w\Delta w(t) - R_{4,u}\big(t; v(t), p(t), w(t)\big)\Big) \end{aligned}$$

Obviously, for given $u \in C(\overline{J}; C(\overline{\Omega}))$, the triple $(v,p,w) \in \mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ is a solution of (4.10)–(4.12) iff $\Psi(u,v,p,w) = 0$ in $L^s(J; \mathbb{L}^p)$. By the assumptions on R_2, R_3 and R_4, Ψ is continuously differentiable and the partial derivative with respect to the second variable in a given point $\big(\bar u, (\bar v, \bar p, \bar w)\big) \in C(\overline{J}; C(\overline{\Omega})) \times \mathrm{MR}^s_0(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ is the linear mapping which assigns to the triple $(h_2, h_3, h_4) \in$

$\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ the expression

$$\left[\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)(h_2,h_3,h_4)\right](t)$$
$$= \left[h_2'(t) - k_v\Delta h_2(t) - \sum_{i=2}^4 \partial_i R_{2,u}\big(t;\bar v(t),\bar p(t),\bar w(t)\big)h_i(t),\right. \tag{4.14}$$
$$h_3'(t) - k_v\Delta h_3(t) - \sum_{i=2}^4 \partial_i R_{3,u}\big(t;\bar v(t),\bar p(t),\bar w(t)\big)h_i(t), \tag{4.15}$$
$$\left. h_4'(t) - k_v\Delta h_4(t) - \sum_{i=2}^4 \partial_i R_{4,u}\big(t;\bar v(t),\bar p(t),\bar w(t)\big)h_i(t)\right], \tag{4.16}$$

which is a function from $L^s(J;\mathbb{L}^p)$. We know already that the operator $-\widetilde{\Delta}$ satisfies maximal parabolic regularity on the space $\mathbb{L}^p$. Moreover, it is clear that the remaining terms in front of the directions h_i in (4.14)–(4.16), considered as time-dependent multipliers on the corresponding L^p-space, form *bounded* operators in $L^s(J;\mathbb{L}^p)$, since the corresponding multipliers are bounded and continuous in space and time. Hence, according to a suitable perturbation theorem as in [9, Prop. 1.3], the equation

$$\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)(h_2,h_3,h_4) = \mathfrak{f}$$

is uniquely solvable for every $\mathfrak{f}\in L^s(J;\mathbb{L}^p)$ with $(h_2,h_3,h_4)\in \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$. This means that the partial derivative $\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)$ is a topological isomorphism between $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ and $L^s(J;\mathbb{L}^p)$, what makes the implicit function theorem applicable. Considering $\Psi(\bar u,\bar v,\bar p,\bar w)=0$ and $(\bar u,\bar v,\bar p,\bar w)=(\bar u,\mathcal{S}(\bar u))$, we thus obtain that the implicit function defined on a neighborhood of $\bar u$, whose existence is guaranteed by the implicit function theorem, coincides with $\mathcal{S}$ on that neighborhood and is continuously differentiable. Since this is true for *every* function $\bar u\in C(\overline{J};C(\overline{\Omega}))$, the "solution operator" $\mathcal{S}$ is continuously differentiable on that space. □

Remark 4.11. In addition to the results of Theorem 4.10, the above considerations make it clear that the set of solutions $\{\mathcal{S}(u)\colon u\in\mathfrak{B}\}$ which corresponds to a *bounded* subset $\mathfrak{B}$ of $C(\overline{J};C(\overline{\Omega}))$ in turn forms a *bounded* subset in the space $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$, and, hence, a precompact one in $C_0(\overline{J};C(\overline{\Omega})^3)$, cf. Lemma 4.5. This can be seen by observing that the real functions $R_{i,u}$, $i=2,3,4$, acting as right hand sides in (4.10)–(4.12) are uniformly bounded in $L^s(J;\mathbb{L}^p)$ in the following way: We set

$$M_R := \max_i M_{i,R} < \infty, \quad \text{where} \quad M_{i,R} := \sup_{\substack{|\bar u|\le M_{\mathfrak{B}},\\ |\bar v|\vee|\bar p|\vee|\bar w|\le M+1}} \big|R_i(\bar u,\bar v,\bar p,\bar w)\big|,$$

using $M_{\mathfrak{B}} := \max_{u\in\mathfrak{B}}\|u\|_{C(\overline{J};C(\overline{\Omega}))}$. Then

$$\max_i \big\|R_{i,u}\big(\cdot\,;v(\cdot),p(\cdot),w(\cdot)\big)\big\|_{L^\infty(J;\mathbb{L}^p)} \le |\Omega|^{\frac1p} M_R$$

for *all* $u\in\mathfrak{B}$ and $(v,p,w)\in\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p))$, which by the maximal parabolic regularity estimate (3.12) shows that $\{\mathcal{S}(u)\colon u\in\mathfrak{B}\}$ forms a bounded set in the space $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$.

Out next intention is to show that the mapping $\mathcal{S}$ is Lipschitzian on bounded subsets of $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$.

Corollary 4.12. *Let $\mathcal{B}$ be any bounded subset of $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. Then the mapping $\mathcal{S}$ is Lipschitzian as a mapping from $\mathcal{B}$ into $\mathrm{MR}^s_0(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$, and hence, also into $C(\overline{J};\mathbb{W}^{1,q})$.*

Proof. Without loss of generality we may assume that $\mathcal{B}$ is a—sufficiently large—ball. Any bounded subset $\mathcal{B}$ of $\mathrm{MR}^s_0(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ forms a precompact subset of $C(\overline{J};C(\overline{\Omega}))$, according to Lemma 4.5. Accordingly, its closure $\overline{\mathcal{B}}$ in $C(\overline{J};C(\overline{\Omega}))$ forms a compact set in this space which is convex, too. Now Theorem 4.10 (ii) tells us that the derivative of $\mathcal{S}$ is bounded on $\overline{\mathcal{B}}$. Since this set contains with any two points also the segment between them, an application of the mean value theorem gives the first claim. Finally, the assertion for $C(\overline{J};\mathbb{W}^{1,q})$ is obtained from the previous one via Lemma 4.5. $\square$

Having introduced the solution operator $\mathcal{S}$ for (4.10)–(4.12), we now turn back to Theorem 4.9. Inserting $\mathcal{S}(u)$ with $u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ for (v,p,w) in (4.9), one obtains a self-consistent equation for u alone together with the initial value condition $u(0)=u_0$. This equation can be solved via Theorem 3.17, as we will show below. Afterwards, having the solution $\bar u$ at hand, the functions $(\bar v,\bar p,\bar w)$ are determined via Lemma 4.10 or $\mathcal{S}(\bar u)$, from which they satisfy (4.10)–(4.12) automatically by construction. The quality of the whole solution of (4.9)–(4.12) is then $\bar u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ and $(\bar v,\bar p,\bar w)\in\mathrm{MR}^s_0(I;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$.

We have formulated the next big step—the application of Theorem 3.17—as a theorem on its own. For this, let $\mathcal{S}_1$ denote the v-component of $\mathcal{S}$, $\mathcal{S}_2$ the p-component of $\mathcal{S}$, and $\mathcal{S}_3$ the w-component of $\mathcal{S}$.

Theorem 4.13. *Suppose $(u_0,v_0,p_0,w_0)\in\mathrm{IV}(r,s)$. Then there exists a maximal interval $I_\bullet=\,]0,S_\bullet[\,\subseteq J$ such that the equation*

$$\begin{aligned} u'(t)&+A\Big(\kappa\big(u(t),v_{\mathcal{I}}(t)+\mathcal{S}_1(u)(t)\big)\Big)u(t)\\ &=A\Big(\sigma\big(u(t),v_{\mathcal{I}}(t)+\mathcal{S}_1(u)(t)\big)\Big)\big(v_{\mathcal{I}}(t)+\mathcal{S}_1(u)(t)\big)+\widehat{R}_1\big(t;u(t),\mathcal{S}(u)(t)\big), \end{aligned}\tag{4.17}$$

has a unique solution $u\in\mathrm{MR}^r(I;\mathrm{dom}_{L^p}(\Delta),L^p)$ with initial value $u(0)=u_0$ on every subinterval $I=\,]0,S[\,\subset I_\bullet$.

In order to validate the suppositions in Theorem 3.17, we will formulate some lemmata:

Lemma 4.14. *Let $\xi\colon\mathbb{R}^2\to\mathbb{R}$ be twice continuously differentiable. Then the superposition operator $(\psi,\varphi)\to\xi(\psi(\cdot),\varphi(\cdot))$ induced by ξ is well defined and Lipschitzian on bounded sets when considered as an operator from $W^{1,q}\times W^{1,q}$ into $W^{1,q}$.*

Proof. Let $\mathcal{B}$ be a bounded set in $W^{1,q}$ and assume firstly that $\psi,\varphi\in\mathcal{B}\cap C^\infty(\Omega)$. Taking into account that $\mathcal{B}$ forms a bounded subset of $C(\overline{\Omega})$, a straight forward calculation shows the existence of a constant $c=c(\mathcal{B},\xi)$ such that

$$\big\|\xi(\psi_1,\varphi_1)-\xi(\psi_2,\varphi_2)\big\|_{W^{1,q}}\le c\big(\|\psi_1-\psi_2\|_{W^{1,q}}+\|\varphi_1-\varphi_2\|_{W^{1,q}}\big),\tag{4.18}$$

holds for all $\psi, \varphi \in \mathcal{B} \cap C^\infty(\Omega)$. Thus, the superposition operator induced by ξ is defined on a dense subset of $\mathcal{B} \times \mathcal{B} \subset W^{1,q} \times W^{1,q}$ and is uniformly continuous in $W^{1,q}$ w.r.t. the $W^{1,q} \times W^{1,q}$-topology. Hence, it can be extended to all of $\mathcal{B} \times \mathcal{B}$, with the same estimate as in (4.18). □

We immediately obtain the following extension from the preceding lemma.

Corollary 4.15. *Let $\xi \colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. In the spirit of Lemma 4.14, ξ induces a superposition operator $C(\overline{J}; W^{1,q}) \times C(\overline{J}; W^{1,q}) \to C(\overline{J}; W^{1,q})$ via*

$$C\big(\overline{J}; W^{1,q}\big) \times C\big(\overline{J}; W^{1,q}\big) \ni (\psi, \varphi) \mapsto \Big[t \mapsto \xi\big(\psi(t), \varphi(t)\big)\Big] \in C\big(\overline{J}; W^{1,q}\big),$$

and this mapping is also Lipschitzian on bounded sets.

The next lemma covers the differential operators occurring in (4.9).

Lemma 4.16. *Let $\xi \colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable.*

i) The operator

$$\mathcal{A}(u)(t) := A\Big(\xi\big(u(t), v_{\mathcal{I}}(t) + \mathcal{S}_1(u)(t)\big)\Big) \tag{4.19}$$

defines a mapping

$$\mathcal{A} \colon \mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p) \to C\big(\overline{J}; \mathcal{L}(\mathrm{dom}_{L^p}(\Delta); L^p)\big).$$

Moreover, $\mathcal{A}$ is Lipschitzian on bounded subsets of $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$.

ii) If, additionally, ξ is a strictly positive function, then $\mathcal{A}(u)|_I$ provides a topological isomorphism between $\mathrm{MR}^r_0(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ and $L^r(I; L^p)$ for every subinterval $I = \,]0, S[\, \subseteq J$ and every $u \in \mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$. In particular, $\mathcal{A}$ satisfies assumptions i) and ii) in Theorem 3.17 for the spaces $X = L^p$ and $Y = \mathrm{dom}_{L^p}(\Delta)$ in this case.

Proof. i) According to Lemma 4.5, both spaces $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ and $\mathrm{MR}^s_0(\mathrm{dom}_{L^p}(\Delta), L^p)$ each embed continuously into $C(\overline{J}; W^{1,q})$. Hence, both u and $\mathcal{S}_1(u)$ are from $C(\overline{J}; W^{1,q})$, cf. Theorem 4.10. Due to to Lemma 4.8 and (4.6), this is also true for the function $v_{\mathcal{I}}(\cdot)$. Thanks to Corollary 4.15, then the function $\xi\big(u(\cdot), v_{\mathcal{I}}(\cdot) + \mathcal{S}_1(u)(\cdot)\big)$ is also from $C(\overline{J}; W^{1,q})$. This allows to apply Corollary 3.23, which shows that $\mathcal{A}$ as given in (4.19), is well-defined as a mapping into the space $C(\overline{J}; \mathcal{L}(\mathrm{dom}_{L^p}(\Delta); L^p))$.

Let us further show the Lipschitz continuity of $\mathcal{A}$ on bounded subsets of the space $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$. Combining Corollary 4.12 and Lemma 4.14 shows that the mapping

$$\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p) \ni u \mapsto \xi\big(u(\cdot), v_{\mathcal{I}}(\cdot) + \mathcal{S}_1(u)(\cdot)\big) \in C(\overline{J}; W^{1,q})$$

is well-defined and Lipschitzian on bounded subset of $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$. Now it remains to apply Corollary 3.23.

ii) Clearly, assumption i) of Theorem 3.17 is already covered by the first assertion in this lemma. Let u be a fixed function from $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$. Under the positivity condition on ξ, the functions $\xi\big(u(t), v_{\mathcal{I}}(\cdot) + \mathcal{S}_1(u)(t)\big) \in W^{1,q}$ are measurable and bounded from above and below by positive constants, uniformly for all $t \in \overline{J}$. Thus, the operators $\mathcal{A}(u)(t)$ satisfy maximal parabolic regularity on L^p for each fixed

$t \in J$, cf. Theorem 3.16. Moreover, $t \mapsto \mathcal{A}(u)(t)$ belongs to $C(\overline{I}; \mathcal{L}(\mathrm{dom}_{L^p}(\Delta); L^p))$ for every subinterval $I = \,]0, S[\, \subseteq J$ by i). But then Theorem 3.15 tells us that the non-autonomous operator $\mathcal{A}(u)$ on every such I satisfies maximal parabolic $L^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$-regularity. This is exactly assumption ii) in Theorem 3.17. $\square$

Let us now turn to the right-hand side in (4.9).

Lemma 4.17. *Define for $u \in \mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ the following operators:*

$$F_1(u) := A\Big(\sigma\big(u(\cdot), v_{\mathcal{I}}(\cdot) + \mathcal{S}_1(u)(\cdot)\big)\Big) v_{\mathcal{I}}(\cdot), \tag{4.20}$$

$$F_2(u) := A\Big(\sigma\big(u(\cdot), v_{\mathcal{I}}(\cdot) + \mathcal{S}_1(u)(\cdot)\big)\Big) \big[\mathcal{S}_1(u)(\cdot)\big], \tag{4.21}$$

$$F_3(u) := \widehat{R}_1\big(\cdot\,; u(\cdot), \mathcal{S}(u)(\cdot)\big). \tag{4.22}$$

Then F_1, F_2 and F_3 are well-defined as mappings from $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ into $L^s(J; L^p)$ and Lipschitzian on bounded sets.

Proof. We first consider F_1 and F_2. Taking $\xi = \sigma$ in Lemma 4.16, we see that the operator function in (4.19) belongs to the space $C(\overline{J}; \mathcal{L}(\mathrm{dom}_{L^p}(\Delta); L^p))$ for every $u \in \mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$. Due to the supposition $v_0 \in (L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}$, cf. (4.5) and (4.6), we already know that in fact $v_{\mathcal{I}} \in L^s(J; \mathrm{dom}_{L^p}(\Delta))$, see Lemma 4.8. For F_2, we recall that $\mathcal{S}_1(u)$ belongs to $L^s(J; \mathrm{dom}_{L^p}(\Delta))$, cf. Theorem 4.10. This shows that F_1 and F_2 are well-defined.

Let us prove the Lipschitz properties for F_1 and F_2. For F_1, this directly follows from Lemma 4.16 with $\xi = \sigma$, and the property $v_{\mathcal{I}} \in L^s(J; \mathrm{dom}_{L^p}(\Delta))$. On the other hand, F_2 is of the form $F_2(u) = \mathcal{A}_\sigma(u)\mathcal{S}_1(u)$, where $\mathcal{A}_\sigma$ is the operator in (4.19) for $\xi = \sigma$, i.e., a product of two functions in u which are Lipschitzian and bounded on bounded sets in $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ with values in the correct spaces, by Lemma 4.16 and Corollary 4.12, see also Remark 4.11. Hence F_2 is also Lipschitzian on bounded sets.

The assertions on F_3 are also satisfied: It remains to collect the continuity of $v_{\mathcal{I}}, p_{\mathcal{I}}$ and $w_{\mathcal{I}}$ due to Lemma 4.8 with the regularity of v_0, p_0 and w_0 as in (4.6), the assumptions on R_1 (cf. Assumption 2.1) and the properties of $\mathcal{S}(\cdot)$ as in Theorem 4.10 combined with Corollary 4.12. $\square$

Lemma 4.18. *Define $\mathcal{A}$ as in* (4.19)*, there setting $\xi := \sigma$. Further, put $F := F_1 + F_2 + F_3$ as given in* (4.20)–(4.22)*. Then both $\mathcal{A}$ and F satisfy the Volterra property, cf. Theorem 3.17.*

Proof. We only need to check the supposition for $\mathcal{S}$. Since $\mathcal{S}(u)$ is obtained as the solution of a system of semilinear parabolic *forward* equations into which u enters *pointwise* with respect to the time variable, it is clear that if $u_1, u_2 \in C(\overline{J}; C(\overline{\Omega}))$ with $u_1 = u_2$ on a subinterval $I = \,]0, S[\, \subseteq J$, then also $\mathcal{S}(u_1)|_I = \mathcal{S}(u_2)|_I$. But this is exactly the Volterra property. $\square$

Now all suppositions of Theorem 3.17 are proved to be satisfied in order to prove Theorem 4.13.

Proof of Theorem 4.13. Since we presupposed the correct regularity for the initial value $u_0 \in (L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{r},r}$, it remains to collect all the assertions from Lemmata 4.16, 4.17 and 4.18. With these, Theorem 3.17 is applicable and, hence, proves Theorem 4.13. $\square$

With Theorem 4.13 at hand, we are now in turn able to prove the main Theorem 4.3 via Theorem 4.9.

Proof of Theorem 4.9. Let $u \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ be the local-in-time solution of (4.17) on an interval $I \subset I_\bullet$ as given by Theorem 4.13. Lemma 4.5 shows that u admits the regularity to obtain $(v,p,w) := \mathcal{S}(u)$ via Theorem 4.10. This proves Theorem 4.9 by construction. □

Proof of Theorem 4.3. We use Theorem 4.9. Let

$$(u, (\check{v}, \check{p}, \check{w})) \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p) \times \mathrm{MR}^s_0(I; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$$

be the solutions of (4.9)–(4.13) as given by Theorem 4.9 (we need to return to the accented way of denoting the functions, as introduced in (4.7), now). It suffices to "remove" the cut-off introduced in Definition 4.7 for $(\check{v}, \check{p}, \check{w})$. Let M be the number from Definition 4.7 for given $\delta > 0$. Firstly, from Lemma 4.8, we know that

$$\|v_\mathcal{I}\|_{C(\overline{I};C(\overline{\Omega}))} \vee \|p_\mathcal{I}\|_{C(\overline{I};C(\overline{\Omega}))} \vee \|w_\mathcal{I}\|_{C(\overline{I};C(\overline{\Omega}))} \leq M.$$

On the other hand, since $\check{v}, \check{p}$ and $\check{w}$ are functions from $C(\overline{I}; C(\overline{\Omega}))$ by Lemma 4.5 with initial value zero, there exists an interval $I_0 = \,]0, S_0[\; \subseteq I$ such that

$$\|\check{v}\|_{C(\overline{I}_0;C(\overline{\Omega}))} \vee \|\check{p}\|_{C(\overline{I}_0;C(\overline{\Omega}))} \vee \|\check{w}\|_{C(\overline{I}_0;C(\overline{\Omega}))} \leq \frac{\delta}{2}.$$

This means that

$$R_j^\eta\big(u(t), v_\mathcal{I}(t) + \check{v}(t), p_\mathcal{I}(t) + \check{p}(t), w_\mathcal{I}(t) + \check{w}(t)\big) \\ = R_j\big(u(t), v_\mathcal{I}(t) + \check{v}(t), p_\mathcal{I}(t) + \check{p}(t), w_\mathcal{I}(t) + \check{w}(t)\big)$$

for every $t \in \overline{I}_0$, hence $(u, (v,p,w))$ with (v,p,w) as in (4.7) are a solution to (4.1)–(4.5) on I_0, cf. (4.8). Moreover, (v,p,w) admit the correct regularity due to $(v_\mathcal{I}, p_\mathcal{I}, w_\mathcal{I}) \in \mathrm{MR}^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$, see Lemma 4.8. □

5. Concluding Remarks

In this concluding chapter we want to comment on possible relaxations and modifications that can be done to apply our results also to some slightly different situations than those that we have proposed in the present paper.

i) Reduction to simplified models: We want to point out again that the simplified model (1.7) may also be treated by the strategy used above for the full model, with very little changes. The same is true for the case of only elliptic equations for v, p, and w, for which one would not need to deal with a nonlocal equation. We refer to the paragraph in the introduction and to [69], where such a system was treated.

ii) Regularity of initial data: We suggest that one can reduce the requirements on the initial values considerably, if one is willing and able to work in spaces with temporal weights. The basis of such an approach are the results in [62] where it is shown that maximal parabolic regularity carries over to spaces with temporal weights. The demanding task would be to prove an analogue of Amann's theorem also in this case and, finally, carry out the program of this paper in that setting. Clearly, this would be an ambitious program and is completely out of scope here.

iii) Boundary conditions in the model: Of course, one can also impose other boundary conditions than homogeneous Neumann conditions. For example, one can also find references where no-flux boundary conditions for the equation of the population density and homogeneous Dirichlet conditions for the chemoattractant, or homogeneous Dirichlet boundary conditions for both equations of the simplified system (1.7) are considered (see for example [25] and [94]). If still other boundary conditions are imposed (as done for instance in [72]) or if the inhomogeneities R_i consist of more delicate terms such as ones "living on the boundary" $\partial\Omega$, one can proceed in a quite similar way, basing on Assumption 3.6 in case of pure Dirichlet conditions or mixed boundary conditions. There also exist large classes of domains for which the assumption is satisfied in these cases, cf. [28]. Then spaces of type $W^{-1,q}$ would be adequate to considering the system in and the principal functional analytical framework would be very similar. In particular, the needed elliptic and parabolic regularity results are also available here, cf. [10, Ch. 11].

iv) Convex domains: In contrast to most known results so far we did not assume the domain Ω to be convex. However, if the domain Ω *is* convex, then it is easier to prove that the Keller-Segel system is well-posed: one is enabled to treat the problem in L^2, basing on the classical result $(-\Delta + 1)^{-1} \colon L^2 \to H^2$, cf. [39, Ch. 3.2]. Namely, from this one deduces

$$\big(L^2, \mathrm{dom}_{L^2}(\Delta)\big)_{\theta,1} \hookrightarrow \big[L^2, \mathrm{dom}_{L^2}(\Delta)\big]_\theta \hookrightarrow \big[L^2, H^2\big]_\theta \hookrightarrow W^{1,4},$$

as long as $\theta \geq \frac{1}{2}(1 + \frac{d}{4})$, the bound on θ being strictly smaller than 1 for space dimensions $d = 2$ or $d = 3$. Thus, one can principally proceed as in our more general proof, thereby avoiding the nontrivial considerations in the non-Hilbert case we used.

v) Regularity of solutions: Concerning the equations for (v,p,w), one could choose any other integrability index $p \in \,]\frac{q}{2}, \infty[$ for the spatial variable. Moreover, it is possible to *bootstrap* the regularity of the solutions by inserting the solutions $(v,p,w) \in \mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p) \hookrightarrow C^\alpha(\overline{J}; C(\overline{\Omega}))$ of (4.10)–(4.12) into the right hand sides, which then each belong to a space $C^\beta(J; C(\overline{\Omega}))$ for some $\beta > 0$. Now exploiting the fact that $-\Delta$ also generates an analytic semigroup on $C(\overline{\Omega})$ (see [79, Rem. 2.6]) and the well known results of [66, Ch. 4], one obtains even more regularity for (v,p,w).

vi) Matrix-valued coefficient functions: Last, we want to point out a technicality concerning our considerations in Chapter 3.1 and 3.2. As already mentioned in the introduction, these considerations may also be generalized to *real matrix-valued* coefficients, that is, the differential operators $-\nabla \cdot \mu\nabla$ where μ is a bounded measurable function on Ω taking its values in the set of positive definite matrices, since the underlying results are available also in this case, cf. [30] and the references therein, see also [28]. We did not undertake this here because the considered Keller-Segel model is restricted to scalar coefficients and the general way to proceed is clear.

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