REGULARIZATION FOR OPTIMAL CONTROL PROBLEMS ASSOCIATED TO NONLINEAR EVOLUTION EQUATIONS

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Abstract. It is well-known that in the case of a sufficiently nonlinear general optimal control problem there is very frequently the necessity for a compactness argument in order to pass to the limit in the state equation in the standard "calculus of variations" proof for the existence of optimal controls. For time-dependent state equations, i.e., evolution equations, this is in particular unfortunate due to the difficult structure of compact sets in Bochner-type spaces. In this paper, we propose an abstract function space $\mathbb{W}^{1,2}_p(X;Y)$ and a suitable regularization- or Tychonov term J_c for the objective functional which allows for the usual standard reasoning in the proof of existence of optimal controls and which admits a reasonably favorable structure in the characterization of optimal solutions via first order necessary conditions in, generally, the form of a variational inequality of obstacle-type in time. We establish the necessary properties of $\mathbb{W}^{1,2}_p(X;Y)$ and J_c and derive the aforementioned variational inequality. The variational inequality can then be reformulated as a projection identity for the optimal control under additional assumptions. We give sufficient conditions on when these are satisfied. The considerations are complemented with a series of practical examples of possible constellations and choices in dependence on the varying control spaces required for the evolution equations at hand.

1. Introduction

In this paper we consider optimal control problems with a separated objective functional in the general abstract form

$$\text{(OCP)}\qquad \begin{aligned} \min_{(y,u)} \quad & \mathsf{J}(y,u) = \mathsf{J}_s(y) + \mathsf{J}_c(u) \\ \text{subject to} \quad & F(y,u) = 0, \\ & u \in \mathcal{U}_{\mathrm{ad}}. \end{aligned}$$

In order to prove existence of optimal solutions to (OCP), it is standard to consider an infimal sequence of feasible points, that is, a sequence $(y_k,u_k)_k \in M_{\mathrm{ad}}$ such that $\lim_{k\to\infty} \mathsf{J}(y_k,u_k) = \inf_{(y,u)\in M_{\mathrm{ad}}} \mathsf{J}(y,u)$, where $M_{\mathrm{ad}} = \{(y,u)\colon F(y,u)=0,\ u\in\mathcal{U}_{\mathrm{ad}}\}$ is the admissible set for (OCP), and to show that a certain subsequence of this infimal sequence converges to the minimizer $(\bar y,\bar u)$. Let us for example assume that the solution mapping $u\mapsto y(u)$, such that $F(y(u),u)=0$, maps bounded sets from the reflexive Banach space of controls $\mathcal{U}$ into bounded sets in the reflexive solution space $\mathcal{Y}$, and that J_c is coercive on $\mathcal{U}$. (We ignore $\mathcal{U}_{\mathrm{ad}}$ for the moment.) Then we obtain a weakly convergent subsequence of $(y_k,u_k)_k$, converging to some $(\bar y,\bar u)$. If now F is nonlinear, then $u\mapsto y(u)$ may in general not directly exhibit the needed weak continuity properties in order to pass to the limit in (OCP), i.e., to show that $F(\bar y,\bar u)=0$. It is usually necessary to use a compactness argument here. In the absence of specialized possibilities such as the div-curl lemma or compensated compactness (cf. [35, 44]), a particular situation is as follows: suppose that, in fact, one has a weaker space $\mathcal{U}_w$ with $\mathcal{U}\hookrightarrow\mathcal{U}_w$ for which one knows strong continuity

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of $\mathcal{U}_w \ni u \mapsto y(u) \in \mathcal{Y}$. This is a common situation where the underlying constraint is well posed also for controls from a weaker space than the control problem is posed in. (We will have a running example throughout the article to illustrate the setting.)

First, let even $\mathcal{U} \hookrightarrow\hookrightarrow \mathcal{U}_w$ (compact embedding). Then we have strong convergence of the weakly convergent subsequence of $(u_k)_k$ in $\mathcal{U}_w$ and can pass to the limit in the state equation. While it is usually not too difficult to identify possible combinations of $\mathcal{U}$ and $\mathcal{U}_w$, the coercive function J_c depends of course strongly on $\mathcal{U}$ and occurs in the first order optimality conditions for (OCP); hence, some care is required.

Another ansatz might be to restrict the problem (OCP) by a condition $u \in \mathcal{C}_{\mathrm{ad}}$ such that $\mathcal{U}_{\mathrm{ad}} \cap \mathcal{C}_{\mathrm{ad}}$ is a compact subset of $\mathcal{U}_w$. Then $(u_k)_k$ has a strongly converging subsequence there and we can again pass to the limit in the state equation. However, the new constraint $u \in \mathcal{C}_{\mathrm{ad}}$ has to be accounted for again in the first order optimality conditions and might be very unpleasant there; in particular, one has to deal with the polar cone of $\mathcal{C}_{\mathrm{ad}}$, and in the case of $\mathcal{U}_{\mathrm{ad}} = \mathcal{U}$, we even have artificially transformed an unconstrained problem to a constrained one. See for instance [25] for an example of this general ansatz where the set $\mathcal{C}_{\mathrm{ad}}$ is ignored in the optimality conditions because of its difficult structure.

In this paper, we want to propose a certain family of suitable function spaces $\mathcal{U}$ in dependence of $\mathcal{U}_w$ and a suitable term J_c in the objective functional. Together they avoid further difficulties in the first order optimality theory as far as possible, in the sense of the derivative of J_c being of favorable structure. Since we view $F(y,u) = 0$ as an operator equation describing an evolution equation over a finite interval J and focus on controls which are (allowed to be) instationary in time, the spaces $\mathcal{U}$ must incorporate time-dependent functions, i.e., be subsets of Bochner-type spaces over J.

The actual spaces are of the form $\mathbb{W}^{1,2}_p(X;Y) := W^{1,2}(J;Y) \cap L^p(J;X)$ for some time interval $J = (T_0, T_1)$, and for reflexive Banach spaces X and Y with $X \hookrightarrow Y$. Further, J_c takes the form

$$\mathsf{J}_c(u) := \int_J \frac{\beta_2}{2} \|\partial_t u(t)\|_Y^2 + \frac{\beta_p}{p} \|u(t)\|_X^p \,\mathrm{d}t \tag{1.1}$$

for some parameters $\beta_2 > 0$ and $\beta_p \geq 0$. While the coupling between $\mathcal{U}_w$ and $\mathcal{U}$ in the explanations above were via an embedding $\mathcal{U} \hookrightarrow\hookrightarrow \mathcal{U}_w$, we in general allow for a coupling by a possibly non-injective operator $\mathcal{E}$. More precisely, we prove in Theorem 2.9 below that if $E \in \mathcal{L}((X,Y)_{\eta,1}; Z)$ for some $\eta \in (0,1)$ such that $E \in \mathcal{K}(X;Z)$, then the time-extension $\mathcal{E}$ defined by $(\mathcal{E}y)(t) := Ey(t)$ maps $\mathbb{W}^{1,2}_p(X;Y)$ *compactly* into $L^r(J;Z)$ for some $r \in [1,\infty]$ or $C^\varrho(J;Z)$ for some $\varrho \in [0,1/2)$, depending on η. (Here $(X,Y)_{\eta,1}$ denotes the real interpolation space between X and Y.).

This gives a huge range of possible, functional-analytically well behaved, spaces $\mathcal{U}_w$ for which $\mathcal{E}$ maps $\mathbb{W}^{1,2}_p(X;Y)$ compactly into $\mathcal{U}_w$. Thereby, in contrast to the common setup of the Aubin-Lions theory [30], we do *not* require the "compactness receiving" space Z to topologically lie between X and Y in the sense of $X \hookrightarrow Z \hookrightarrow Y$. This is of practical importance in the setup of many optimal control problems, cf. the running example throughout the paper and §4.4.

As another central point, the space Y should be chosen as a Hilbert space in order to obtain a derivative $\mathsf{J}_c'(u)$ depending *linearly* on $\partial_t u$. (We will have to require X to be smooth also, of course.) This is in fact the main motivation to use the spaces $\mathbb{W}^{1,2}_p(X;Y)$ instead of the well-known spaces $\mathbb{W}^{1,p}_p(X;Y)$ or generally $\mathbb{W}^{1,q}_p(X;Y)$ for $q \neq 2$ since it allows to obtain a more reasonable characterization of a designated optimal control than in the case $q \neq 2$. More precisely, the Hilbert space structure of Y together with the square in J_c will make it possible to apply an integration by parts formula in the optimality conditions of (OCP) which then yields a second-order Banach-space differential equation or variational inequality, depending on the constraint set $\mathcal{U}_{\mathrm{ad}}$. We will return to these considerations in §3. In the case of a variational inequality, we further obtain a pointwise projection description of the optimal control $\bar{u}$ under additional assumptions. These assumptions are nontrivial and we devote §3.4 to their verification in certain cases. The projection formula could then be a possible starting point for numerical algorithms.

Optimal control problems subject to nonlinear parabolic equations on Lipschitz domains were the starting point of the investigations cf. [20, 23, 24] or [32, 33, 45] for further references, and the reader may imagine having such a problem as the practical incarnation of F. However, the theory presented in this work is not necessarily restricted to parabolic problems, but to time-dependent problems, i.e., evolution equations, in general.

1.1. **Context and related work.** Let us give a brief overview over the context of our work: Results in the spirit of $W^{1,q}(J;Y) \cap L^p(J;X) \hookrightarrow\hookrightarrow L^r(J;Z)$ under appropriate relations and a compactness assumption between X, Y and Z are classical ever since the famous paper of Aubin from 1963 [4]. There, the assumption for the spaces is $X \hookrightarrow\hookrightarrow Z \hookrightarrow Y$, and it is shown that if $\mathcal{B}$ denotes a bounded set in $L^p(J;X)$ for which $\{\partial_t f \colon f \in \mathcal{B}\}$ is bounded in $L^q(J;Y)$, then $\mathcal{B}$ is relatively compact in the space $L^q(J;Z)$. Usage of this result which became known as the *Aubin-Dubinskii lemma* was widely popularized by Lions [30] who used it to great success in the treatment of nonlinear partial differential equations. (Dubinskii proved an analogous, more general version of Aubins result in [15] in 1965). In [43], Simon gave a complete characterization of compact sets in $L^r(J;Z)$ spaces quite in analogy to the Arzelà-Ascoli theorem, and he used this to sharpen Aubin's result by replacing the assumption for $\{\partial_t f \colon f \in \mathcal{B}\}$ to be bounded in $L^q(J;Y)$ by $\|\tau_h f - f\|_{L^q(T_0,T_1-h;Y)} \to 0$ uniformly for $f \in \mathcal{B}$ as $h \searrow 0$, where τ_h denotes the time translation $f \mapsto f(\cdot+h)$, cf. [43, Thm. 5]. There is also a result concerning compactness in $C(J;Z)$, see [43, Cor. 8]. Simon's results in turn were improved and sharpened by many; for example by Amann, who imposes a slightly more restrictive assumption on the spaces in [3]: There, $X \hookrightarrow\hookrightarrow Y$ and $(X,Y)_{\theta,1} \hookrightarrow Z \hookrightarrow Y$ is supposed, for some $\theta \in (0,1)$, and the time translations have to go to zero at a certain rate. (Recall that $X \hookrightarrow (X,Y)_{\theta,1}$ in any case.) Moreover, [3] significantly generalizes the considered spaces, including switching to spaces defined on bounded open sets $\Lambda \subset \mathbb{R}^n$ which satisfy the extension property, cf. [3, §7], see also §4.1 below. Still, the main drive behind the development of all these very helpful devices is the treatment of nonlinear partial differential equations, for which the requirement $X \hookrightarrow Z \hookrightarrow Y$ is evident and necessary. In contrast, we plan to apply a similar idea to the control space in an optimal control setting, which does *not* necessarily need the spaces to be nested hierarchically. In this sense, the designated space in which compactness is needed need not lie *between* X and Y and there needs to be no direct relation to Y itself, only to its interpolation spaces with X.

The need to employ the $\mathbb{W}^{1,2}_p(X,Y)$ in the first place arises from insufficient regularization properties of classical Hilbert space Tikhonov terms in the objective functional, if the underlying PDE (system) is strongly nonlinear. As explained above, this results in the inability to pass to the limit in the PDE starting from a minimizing sequence for the objective functional due to the lack of weak continuity. This occurs in particular in coupled systems of nonlinear PDEs which are often necessary to have for a realistic modeling of complex dynamical systems. See the running example throughout the paper or again [32, 33] for actual examples. From the authors' point of view, the need to have a comprehensive Banach space optimal control theory for such strongly nonlinear equations arises.

We are not aware of works so far which investigate the $\mathbb{W}^{1,2}_p(X,Y)$ spaces or comparable ones in a general optimal control setting.

1.2. **Organization.** The paper is organized as follows: First, we fix some basic notation and definitions. In the next section, we provide a suitable functional analytic groundwork, establishing general compactness properties for spaces of the form $\mathbb{W}^{1,2}_p(X;Y)$ under appropriate assumptions. This is the basic property as explained in the introduction. In §3, related aspects of the control problems are discussed: We consider existence of optimal solutions to (OCP) in §3.1 and the impact of the special form of J_c in the optimality conditions in §3.3. There, we will see that the

Hilbert space structure for Y allows to derive an explicit Banach-space differential equation for the second derivative of the designated optimal control, and a corresponding variational inequality in the case of control constraints. The considerations are complemented with a brief detour regarding smoothness and duality mappings Banach spaces in §3.2 and an additional subsection regarding time regularity of the optimal control in a control-constrained setting, §3.4. With this time regularity, one may reformulate the variational inequality characterizing a locally optimal control to a projection identity. In §4 we show how to choose X, Y and the integrability exponent p for a list of concrete examples of control spaces $\mathcal{U}_w$.

1.3. **Basic notation and definitions.** In the following, $J = (T_0, T_1)$ denotes a generic non-empty interval with $-\infty < T_0 < T_1 < \infty$. All Banach spaces under consideration are supposed to be real. We follow the general convention that caligraphic letters like $\mathcal{U}$ or $\mathcal{Y}$ etc. always stand for function spaces on J with values in another Banach space (or subsets of such spaces), whereas the standard letters U, Y, X are used for "spatial" function spaces on a domain $\Lambda \subset \mathbb{R}^n$, or as placeholders for general Banach spaces of whatever kind.

For two Banach spaces X and Y, we say that Y is (continuously) embedded into X, abbreviated by $X \hookrightarrow Y$, if $Y \subseteq X$ and the identity mapping from Y into X is continuous. Analogously, $Y \hookrightarrow\hookrightarrow X$ stands for Y being compactly embedded into X, where $Y \subseteq X$ and the identity mapping from Y to X is compact. Bounded linear operators between Banach spaces are denoted by $\mathcal{L}(X; Y)$, and the subspace of compact linear operators by $\mathcal{K}(X; Y)$. Further, we denote by $L^r(J; X)$ the set of all Bochner-measurable functions w, for which the function $J \ni t \mapsto \|w(t)\|_X$ belongs to $L^r(J)$ with the corresponding norm, for $1 \leq r \leq \infty$. The subspace of all functions from $L^r(J; X)$ possessing a derivative in the sense of X-valued distributions also belonging to $L^r(J; X)$ is called $W^{1,r}(J; X)$, cf. [10, Ch. XVIII.1.1] or [2, Ch. III.1] and we give it the norm $\|f\|_{W^{1,r}(J;X)} = \|\partial_t f\|_{L^r(J;X)} + \|f\|_{L^r(J;X)}$. Vector-valued Hölder-spaces are denoted by $C^\varrho(J; X)$ for $\varrho \in (0, 1)$ with the Hölder norm, see [2, Ch. II.1.1], where we always identify a Hölder-continuous function on J with its unique extension to $\overline{J}$. We sometimes also write $C^\varrho(J; X)$ with the range $\varrho \in [0, 1)$ – in this case, $C(J; X) = C^0(J; X)$ is to be read als the space of (uniformly) continuous functions on $\overline{J}$, equipped with the supremum-norm.

For Banach spaces X, Y and for $\theta \in (0, 1)$ and $\tau \in [1, \infty]$, $(X, Y)_{\theta,\tau}$ denotes the corresponding real interpolation space, and $[X, Y]_\theta$ the corresponding complex interpolation space, cf. [46, Ch. 1], see also [2, Ch. I.2.4].

Finally, by C we denote a generic positive constant.

2. Compactness in Bochner-Lebesgue and (Hölder-)continuous spaces

We first review the two most important compactness characterizations for Bochner-type spaces: The Fréchet-Kolmogorov-Simon theorem for compactness in the $L^q(J; F)$ scale and of course the Arzelà-Ascoli theorem for compactness in the space of continuous functions $C(J; F)$.

Let us assume that X and Y are Banach spaces with $X \hookrightarrow Y$ densely, and let $J = (T_0, T_1)$ be a finite interval. As announced in the introduction, we set

$$\mathbb{W}^{1,2}_p(X; Y) = W^{1,2}(J; Y) \cap L^p(J; X)$$

with the usual intersection norm

$$\|y\|_{\mathbb{W}^{1,2}_p(X;Y)} = \|y\|_{W^{1,2}(J;Y)} + \|y\|_{L^p(J;X)}.$$

Remark 2.1. Up to equivalence of norms, we have

$$\mathbb{W}^{1,2}_p(X; Y) \doteq \dot{W}^{1,2}(J; Y) \cap L^p(J; X),$$

where the latter is the set of all functions $\{f \in L^p(J; X) \colon \partial_t f \in L^2(J; Y)\}$ equipped with the norm $f \mapsto \|\partial_t f\|_{L^2(J;Y)} + \|f\|_{L^p(J;X)}$. This follows from $\dot{W}^{1,2}(J; Y) \cap L^p(J; X) \hookrightarrow C(J; Y)$, see e.g. [46, Lem. 1.8.1], cf. also [3, Lem. 6.1].

We first state the Fréchet-Kolmogorov-Simon theorem [43, Thm. 1]:

Theorem 2.2 (Fréchet-Kolmogorov-Simon)**.** *Let F be a Banach space, and let $\Phi \subset L^q(J;F)$ for some $1 \le q \le \infty$. Then Φ is relatively compact in $L^q(J;F)$ if $q \in [1,\infty)$, respectively in $C(J;F)$ if $q = \infty$, if and only if*

(i) $\|f(\cdot+h) - f(\cdot)\|_{L^q(T_0,T_1-h;F)} \to 0$ *as* $h \to 0$ *uniformly in* $f \in \Phi$,
(ii) $\left\{\int_s^t f(\tau)\,\mathrm{d}\tau \colon f \in \Phi\right\}$ *is relatively compact in F for all $s,t \in J$ with $s < t$.*

One can see rather clearly a division of the requirements in Theorem 2.2 into a *time regularity* assumption, so the first one, and a *"spatial" compactness* assumption in the second one. This observation will be of particular interest to us because it already shows that the elements of a compact set in a Bochner-Lebesgue space must in fact exhibit a better type of time regularity than this class of functions generally admits.

The particular form of Theorem 2.2 which we employ is the following simple modification of [43, Cor. 8]:

Proposition 2.3. *Let $\Phi \subset L^p(J;X)$. Let further Z be another Banach space and suppose that $E \in \mathcal{K}((X,Y)_{\tau,1};Z)$ for some $\tau \in (0,1)$. Set $\mathcal{E}$ by $(\mathcal{E}y)(t) := E(y(t))$ and assume that the following conditions are satisfied:*

(i) *The derivatives $\{\partial_t f \colon f \in \Phi\}$ are bounded in $L^2(J;Y)$,*
(ii) *Φ is bounded in $L^p(J;X)$.*

Then $\mathcal{E}\Phi$ is a compact subset of $L^r(J;Z)$ for every $\frac{1}{r} > \frac{1-\tau}{p} - \frac{\tau}{2}$ if $\tau \le \frac{2}{2+p}$, and a compact subset of $C(J;Z)$ if $\tau > \frac{2}{2+p}$.

Let us point out that the space Z itself need *not* lie between X and Y, so it needs not be embedded into Y in particular. This is the small but, for certain setups very useful, difference between Proposition 2.3 and the quoted classical result of Simon and others [43, Cor. 8]. See the running example and §4.4.

While Theorem 2.2 already gives an assertion about compactness in the space of continuous functions, the classical complete characterization is given by the vector-valued Arzelà-Ascoli theorem [26, Thm. 3.1]:

Theorem 2.4 (Arzelà-Ascoli)**.** *Let K be a compact subset of a metric space, and let F be a Banach space. Let Φ be a subset of the space of continuous functions $C(K;F)$ with the supremum-norm. Then Φ is relatively compact in $C(K;F)$ if and only if the following two conditions are satisfied:*

(i) *Φ is equicontinuous,*
(ii) *for each $x \in K$, the set $\{f(x) \colon f \in \Phi\}$ is relatively compact in F.*

Again, we have a distinction between temporal regularity (equicontinuity) and spatial compactness. We use the theorem in the following way:

Corollary 2.5. *Let Z be a Banach space, let $0 \le \alpha < \omega < 1$, and suppose $E \in \mathcal{K}((X,Y)_{\tau,1};Z)$ for some $\tau \in (0,1)$. Then the linear operator $\mathcal{E} \colon C^\omega(J;(X,Y)_{\tau,1}) \to C^\alpha(J;Z)$ defined by $(\mathcal{E}y)(t) := E(y(t))$ is* compact.

Finally, the connection between the spaces of (Hölder)-continuous functions and $\mathbb{W}^{1,2}_p(X;Y)$ is as follows:

Theorem 2.6. *Let $p \in (1,\infty)$ and let $\tau \in (\frac{2}{2+p}, 1)$. Then we have*

$$\mathbb{W}^{1,2}_p(X;Y) \hookrightarrow C^\varrho(J;(X,Y)_{\tau,1})$$

for $\varrho = \varrho(\tau) = \frac{\tau}{2} - \frac{1-\tau}{p}$.

We give the proof of Theorem 2.6 because it is remarkably elementary with a little knowledge of interpolation theory. Let us recall the following embedding result which follows immediately from the construction of real interpolation spaces by means of the trace method, see [46, Ch. 1.8.3] and [2, Thm. III.4.10.2]:

Lemma 2.7. *Let $p \in (1,\infty)$ and $\theta = \frac{2}{p+2}$. Then we have $\mathbb{W}^{1,2}_p(X;Y) \hookrightarrow C(J;(X,Y)_{\theta,\frac{1}{\theta}})$.*

The proof of Theorem 2.6 now mainly consists of interpolating this embedding with the well known embedding $W^{1,2}(J;Y) \hookrightarrow C^{1/2}(J;Y)$, cf. [3, Ch. 3]. This also underlines why there is the natural upper bound of $1/2$ for ϱ in Theorem 2.6 (for $\tau \nearrow 1$).

Proof of Theorem 2.6. From the $W^{1,2}(J;Y) \hookrightarrow C^{1/2}(J;Y)$ embedding we have, for $t,s \in J$ and $u \in \mathbb{W}^{1,2}_p(X;Y)$, the Hölder estimate

$$\|u(t)-u(s)\|_Y = \left\| \int_s^t \partial_t u(r)\,\mathrm{d}r \right\|_Y \leq |t-s|^{1/2} \left(\int_J \|\partial_t u(r)\|_Y^2\,\mathrm{d}r \right)^{1/2} \tag{2.1}$$

holds true. Set $\theta := \frac{2}{2+p}$. By the reiteration theorem (cf. [46, Ch. 1.10.2]), we write $(X,Y)_{\tau,1}$ as $((X,Y)_{\theta,\frac{1}{\theta}},Y)_{\lambda,1}$ with $\lambda = \frac{\tau-\theta}{1-\theta}$. This allows to deduce the following Hölder estimate:

$$\frac{\|u(t)-u(s)\|_{(X,Y)_{\tau,1}}}{|t-s|^{\frac{\lambda}{2}}} \leq \left(\|u(t)\|_{(X,Y)_{\theta,\frac{1}{\theta}}} + \|u(s)\|_{(X,Y)_{\theta,\frac{1}{\theta}}}\right)^{1-\lambda} \frac{\|u(t)-u(s)\|_Y^\lambda}{|t-s|^{\frac{\lambda}{2}}}$$

for all $t,s \in J$. Now, employing Lemma 2.7 and (2.1), one obtains

$$\frac{\|u(t)-u(s)\|_{(X,Y)_{\tau,1}}}{|t-s|^{\frac{\lambda}{2}}} \leq C\|u\|_{\mathbb{W}^{1,2}_p(X;Y)}$$

and the embedding $C(J;(X,Y)_{\theta,\frac{1}{\theta}}) \hookrightarrow C(J;(X,Y)_{\tau,1})$ if $\tau > \theta$, cf. [46, Ch. 1.3.3 (4)] together with Lemma 2.7 implies the assertion with $\varrho = \lambda/2$. ☐

Before we prove our main theorem for this section, we recall the following compactness propagation property for interpolation scales. It follows from [46, Ch. 1.16.4, Thm. 1] and the reiteration theorem ([46, Ch. 1.10.2]).

Lemma 2.8. *Let Z be a Banach space and let $E \in \mathcal{L}((X,Y)_{\eta,1};Z)$ for some $\eta \in (0,1)$ with $E \in \mathcal{K}(X;Z)$. Then $E \in \mathcal{K}((X,Y)_{\tau,1};Z)$ for all $\tau \in (0,\eta)$.*

Theorem 2.9. *Let $p \in (1,\infty)$ and let X,Y and Z be Banach spaces where $X \hookrightarrow Y$ densely. Suppose that for $\eta \in (0,1)$ there is a continuous linear operator $E \in \mathcal{L}((X,Y)_{\eta,1};Z)$ such that $E \in \mathcal{K}(X;Z)$. Then the linear operator defined by $(\mathcal{E}y)(t) := E(y(t))$,*

$$\mathcal{E}\colon \quad \mathbb{W}^{1,2}_p(X;Y) \to \begin{cases} L^r(J;Z) & \text{for } \frac{1}{r} > \frac{1-\eta}{p} - \frac{\eta}{2} \quad \text{if } 0 < \eta \leq \frac{2}{2+p}, \\ C^\varrho(J;Z) & \text{for } 0 \leq \varrho < \frac{\eta}{2} - \frac{1-\eta}{p} \quad \text{if } \frac{2}{2+p} < \eta < 1, \end{cases}$$

is compact.

Proof. By Lemma 2.8, we have $E \in \mathcal{K}((X,Y)_{\tau,1};Z)$ for all $\tau \in (0,\eta)$. The assertions now follow from Proposition 2.3 for the $L^r(J;Z)$ embedding and from Theorem 2.6 together with Corollary 2.5 for the (Hölder-)continuous scale. Note that the ranges for r and ϱ are non-strictly restricted with respect to the interpolation parameter, so there is no loss passing from η to $\tau \in (0,\eta)$. ☐

Remark 2.10. Let us mention that the proof of Theorem 2.9 can also be done by using the results of Amann [3], for instance by establishing an embedding of $\mathbb{W}^{1,2}_p(X;Y)$ in a Bochner-Sobolev-Slobodeckij space with values in an interpolation space and then using compactness results for these kind of spaces [3, Sect. 5]. Since the results in [3] are quite involved, we have decided to give a different, somewhat more superficial exposition with a straightforward proof of Theorem 2.6. A quite useful observation however is that, quite in analogy to Theorem 2.6, we have

$$\mathbb{W}^{1,2}_p(X;Y) \hookrightarrow L^s(J;(X,Y)_{\eta,1}) \quad \text{where} \quad \frac{1}{s} > \frac{1-\eta}{p} - \frac{\eta}{2}$$

as we infer from [3, Thm. 5.2]. This can also be used to re-obtain Proposition 2.3 by applying [43, Cor. 8] to the set Φ as a subset of $L^s(J;(X,Y)_{\eta,1})$. Doing so mimics the line of thought used for the case of (Hölder-) continuous spaces.

3. Related aspects in the optimal control problem

We return to the optimal control problem

$$\text{(OCP)} \qquad \begin{aligned} \min_{(y,u)} \quad & \mathsf{J}(y,u) = \mathsf{J}_s(y) + \mathsf{J}_c(u) \\ \text{subject to} \quad & F(y,u) = 0, \\ & u \in \mathcal{U}_{\mathrm{ad}}. \end{aligned}$$

We assume that for every $u \in \mathcal{U}_w$, there exists a unique $y \in \mathcal{Y}$ such that $F(y,u) = 0$, that is, the so-called *control-to-state operator* $\mathcal{U}_w \ni u \mapsto y(u) \in \mathcal{Y}$ such that $F(y(u),u) = 0$ is well-defined. Here, $F\colon \mathcal{Y} \times \mathcal{U}_w \to \mathcal{Z}$ is meant as a nonlinear evolution equation with possibly additional constraints. The set $\mathcal{U}_{\mathrm{ad}}$ stands for possible constraints on the controls u.

We give an example of an evolution problem modeled by F with lack of compactness for the control-to-state operator on $\mathcal{U}_w$ or $\mathcal{U}_{\mathrm{ad}}$. This example will be used and continued throughout the article to illustrate the results.

Example. Let us consider the following PDE system as a model example:

$$\begin{aligned} \partial_t \theta - \Delta\theta &= |\nabla\varphi|^2 \quad \text{in } \Omega, & \nabla\theta\cdot\nu &= \theta_\ell \quad \text{on } \partial\Omega, & \theta(0) &= 0, \\ -\Delta\varphi &= 0 \quad \text{in } \Omega, & \nabla\varphi\cdot\nu &= u \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

which is to be satisfied on the time interval J and an underlying (sufficiently smooth) spatial domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$. This is a setup for a boundary control problem for a nonlinearly coupled state system, and this particular version is a modification of a so-called thermistor problem. (See e.g. [25, 32, 33] for a treatment of much more involved versions.) We suppose that $\theta_\ell \in L^\infty(J;L^\infty(\partial\Omega))$. The abstract model for $y = (\theta,\varphi)$ is

$$F(y,u) = \begin{pmatrix} \partial_t\theta - \Delta\theta - |\nabla\varphi|^2 - \theta_\ell \\ -\Delta\varphi - u \end{pmatrix} \tag{3.2}$$

with

$$\mathcal{Y} = \Big(W^{1,s}_0\big(J;W^{-1,q}_0(\Omega)\big) \cap L^r\big(J;W^{1,q}(\Omega)\big)\Big) \times L^{2s}\big(J;W^{1,q}(\Omega)\big)$$

and

$$\mathcal{Z} = L^s\big(J;W^{-1,q}_0(\Omega)\big) \times L^{2s}\big(J;W^{-1,q}_0(\Omega)\big),$$

where $W^{-1,q}_0(\Omega)$ is the dual space of $W^{1,q'}(\Omega)$. Due to the inhomogeneous Neumann boundary data, we treat the system in a weak setting. In accordance with $\mathcal{Z}$, the weak control space would be $\mathcal{U}_w = L^{2s}(J;W^{-1,q}_0(\Omega))$. Under the condition $q > d$ there is a well defined and continuous control-to-state operator mapping from $\mathcal{U}_w$ to $\mathcal{Y}$ for $s \in (1,\infty)$. This can be proven as in [25]. However, due to the nonlinear coupling in the state equation, the control-to-state operator is *not* weakly continuous on $\mathcal{U}_w$. Hence, with only weak convergence in $\mathcal{U}_w$ for u, we cannot pass to the limit in the PDE system.

The situation does not improve if we incorporate e.g. almost everywhere pointwise control bounds for a weakly convergent sequence of controls (u_k). These are only reasonable for objects with a meaningful pointwise (almost everywhere) meaning, like functions in a space of type $L^\infty(J;L^t(\partial\Omega))$, and we would incorporate that space the definition of F via $\operatorname{tr}^* u$ instead of u, where tr^* is the adjoint of the trace operator tr for $\partial\Omega$. This operator $E = \mathrm{tr}^*$ maps $L^t(\partial\Omega)$ to $W^{-1,q}(\Omega)$ for t large enough. (See §4.4 for details.) Hence $\mathcal{E}$ derived from E maps $L^\infty(J; L^t(\partial\Omega))$ to $\mathcal{U}_w$, but *not* compactly, since the image set lacks time regularity, recall Theorem 2.2, see also [43, Prop. 2]. Hence, even pointwise control constraints which give rise to a bounded set in $L^\infty(J; L^\infty(\partial\Omega))$ would not help to gain compactness here.

Let us now turn to the actual assumptions which allow to use a control space of type $\mathbb{W}^{1,2}_p(X;Y)$ in order to achieve a satisfactory theory:

Assumption 3.1. *The following properties hold true:*

(i) *The control-to-state operator* $\mathcal{U}_w \ni u \mapsto y(u) \in \mathcal{Y}$ *is continuous, but* not *assumed to be weakly continuous in any sense,*
(ii) *we have* $\mathcal{U}_w = L^r(J;U)$ *for some* $1 \le r \le \infty$ *or* $\mathcal{U}_w = C^\varrho(J;U)$ *for some* $\rho \in [0,1/2)$, *with a Banach space* U,
(iii) *there are reflexive Banach spaces* X, Y *with* $X \hookrightarrow Y$ *densely and a linear operator* E *with an* $\eta \in (0,1)$ *such that* $E \in \mathcal{L}((X,Y)_{\eta,1};U)$ *and* $E \in \mathcal{K}(X;U)$, *where* $\varrho < \frac{\eta}{2} - \frac{1-\eta}{p}$ *if* $\eta > \frac{2}{2+p}$, *and* $\frac{1}{r} > \frac{1-\eta}{p} - \frac{\eta}{2}$ *otherwise,*
(iv) $\mathcal{U}_{\mathrm{ad}}$ *is nonempty, closed and convex (and hence weakly closed) in* $\mathbb{W}^{1,2}_p(X;Y)$,
(v) $\mathsf{J}_s\colon \mathcal{Y} \to \mathbb{R}$ *is lower semicontinuous and* J_s *is bounded from below over* $y(\mathcal{E}\mathcal{U}_{\mathrm{ad}})$,
(vi) $\mathsf{J}_c\colon \mathbb{W}^{1,2}_p(X;Y) \to \mathbb{R}$ *is given as in* (1.1).

By the assumptions, Theorem 2.9 tells us that $\mathcal{E}$ maps $\mathbb{W}^{1,2}_p(X;Y)$ compactly to $\mathcal{U}_w$, where again (and from now on) $(\mathcal{E}y)(t) = Ey(t)$. Using the control-to-state operator and $\mathcal{E}$, the problem (OCP) may equivalently be reduced to the control u incorporating the space $\mathbb{W}^{1,2}_p(X;Y)$, which results in the following definition:

Definition 3.2. We call the following optimization problem the *reduced optimal control problem*:

$$(\mathrm{OCP}_u) \qquad \min_u \quad j(u) = \mathsf{J}(y(\mathcal{E}u), u) \qquad \text{subject to} \quad u \in \mathcal{U}_{\mathrm{ad}}.$$

As explained in the introductionary chapter, the main motivation to use the spaces $\mathbb{W}^{1,2}_p(X;Y)$ stems from the compactness property as in Theorem 2.9, together with a certain well-behavedness with respect to further uses in optimality theory if J_c as defined in the introduction (see also (1.1) below) is used in the objective functional. We intend to lay out these benefits in this chapter. More precisely, we show that J_c admits the usual properties needed to show existence of optimal solutions to (OCP_u), and that the choice of Y as a Hilbert space allows to derive a concise characterization of locally optimal solutions to (OCP_u).

3.1. **Existence of optimal controls.** As already pointed out in the introduction, the term corresponding to $\mathbb{W}^{1,2}_p(X;Y)$ to be used in the objective functional is

$$\mathsf{J}_c(u) := \int_J \frac{\beta_2}{2}\|\partial_t u(t)\|_Y^2 + \frac{\beta_p}{p}\|u(t)\|_X^p \,\mathrm{d}t \tag{1.1}$$

for some $\beta_2 > 0$ and $\beta_p \ge 0$. For the following, let the assumptions in Assumption 3.1 hold true.

We now collect some properties of J_c and $\mathbb{W}^{1,2}_p(X;Y)$. Note that while the following results are also valid for $p \in (1,2)$, in that case the space $\mathbb{W}^{1,2}_2(X;Y) \hookrightarrow \mathbb{W}^{1,2}_p(X;Y)$ would also admit all needed properties and embeddings and be of easier structure than $\mathbb{W}^{1,2}_p(X;Y)$.

Lemma 3.3. *The spaces* $\mathbb{W}^{1,2}_p(X;Y)$ *are reflexive for every* $p \in (1,\infty)$.

Proof. The spaces $L^p(J;X)$ and $L^2(J;Y)$ are reflexive since X and Y are reflexive (see [12, Cor. IV.1.1]). Then $W^{1,2}(J;Y)$ is also reflexive, since it is isometrically isomorphic to a closed subspace of $L^2(J;Y) \times L^2(J;Y)$ via $u \mapsto (\partial_t u, u)$. Hence $\mathbb{W}^{1,2}_p(X;Y)$ is reflexive as an intersection of reflexive spaces with $W^{1,\max(2,p)}(J;Y)$ acting as a common (Hausdorff) superspace. □

Lemma 3.4. *The function* J_c *is weakly lower semicontinuous on* $\mathbb{W}^{1,2}_p(X;Y)$.

Proof. Clearly, both $u \mapsto \|\partial_t u\|^2_{L^2(J;Y)}$ and $u \mapsto \|u\|^p_{L^p(J;X)}$ are continuous convex functions on $\mathbb{W}^{1,2}_p(X;Y)$, hence J_c is also continous and convex and as such weakly lower semicontinuous. ☐

Lemma 3.5. *Let* $\beta_p > 0$. *Then the function* J_c *is coercive on* $\mathbb{W}^{1,2}_p(X;Y)$.

Proof. Let (u_k) be a sequence in $\mathbb{W}^{1,2}_p(X;Y)$ such that $\|u_k\|_{\mathbb{W}^{1,2}_p(X;Y)} \to \infty$ as $n \to \infty$. We show that $\mathsf{J}_c(u_k) \to \infty$, too. Due to Remark 2.1, we have $\|u\|_{\mathbb{W}^{1,2}_p(X;Y)} \lesssim \|\partial_t u\|_{L^2(J;Y)} + \|u\|_{L^p(J;X)}$ for all $u \in \mathbb{W}^{1,2}_p(X;Y)$. Hence, $\|\partial_t u_k\|_{L^2(J;Y)} + \|u_k\|_{L^p(J;X)} \to \infty$ as $k \to \infty$ as well. Thus, at least one of the summands must go to infinity, say, $\|u_k\|_{L^p(J;X)}$. But this implies that $\|u_k\|^p_{L^p(J;X)} \to \infty$ and accordingly $\mathsf{J}_c(u_k) \to \infty$ since J_c is bounded from below by zero. The case where $\|\partial_t u_k\|_{L^2(J;Y)}$ goes to infinity works analogously. ☐

Lemma 3.6. *The control-to-state operator* $u \mapsto y_{\mathcal{E}}(u) := y(\mathcal{E}u)$ *is weak-strong continuous from* $\mathbb{W}^{1,2}_p(X;Y)$ *to* $\mathcal{Y}$.

Proof. Let $u_k \rightharpoonup \bar{u}$ in $\mathbb{W}^{1,2}_p(X;Y)$. Then, by Theorem 2.9, $(\mathcal{E}u_k)$ is strongly convergent in $\mathcal{U}_w$ to $\mathcal{E}\bar{u}$. Since $\mathcal{U}_w \ni u \mapsto y(u) \in \mathcal{Y}$ was continuous by Assumption 3.1 (i), we find that $y_{\mathcal{E}}(u_k) \to y_{\mathcal{E}}(\bar{u})$, i.e., $\mathbb{W}^{1,2}_p(X;Y) \ni u \mapsto y_{\mathcal{E}}(u) \in \mathcal{Y}$ is weak-strong continuous. ☐

Theorem 3.7. *Let* $\beta_p > 0$ *or let* $\mathcal{U}_{\mathrm{ad}}$ *be bounded in* $L^p(J;X)$. *Then there exists an optimal solution* $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ *to* (OCP_u).

Proof. Since there exists a feasible point for (OCP_u), we consider an infimal sequence $(u_k) \subset \mathcal{U}_{\mathrm{ad}}$ such that $j(u_k) \to \inf_{u \in \mathcal{U}_{\mathrm{ad}}} j(u)$. As j is bounded from below over $\mathcal{U}_{\mathrm{ad}}$, the sequences $(\mathsf{J}_s(y_{\mathcal{E}}(u_k)))$ and $(\mathsf{J}_c(u_k))$ must be bounded. If $\beta_p > 0$, then by Lemma 3.5, boundedness of $\mathsf{J}_c(u_k)$ implies boundedness of (u_k) in $\mathbb{W}^{1,2}_p(X;Y)$. If on the other hand $\beta_p = 0$ but $\mathcal{U}_{\mathrm{ad}}$ is bounded in $L^p(J;X)$, then (u_k) is overall again bounded in $\mathbb{W}^{1,2}_p(X;Y)$. Now using Lemma 3.3 gives us a subsequence (u_{k_ℓ}) that converges weakly in $\mathbb{W}^{1,2}_p(X;Y)$ to some $\bar{u}$. The set $\mathcal{U}_{\mathrm{ad}}$ is weakly closed, hence $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$. Moreover, Lemma 3.6 shows that $(y_{\mathcal{E}}(u_{k_\ell}))$ converges strongly in $\mathcal{Y}$. Finally, j is weakly lower semicontinuous on $\mathbb{W}^{1,2}_p(X;Y)$, see Lemma 3.4 and the assumption on J_s, such that we find $\inf_{u \in \mathcal{U}_{\mathrm{ad}}} j(u) = \lim_{\ell \to \infty} j(u_{k_\ell}) \geq j(\bar{u})$, i.e., $j(\bar{u}) = \inf_{u \in \mathcal{U}_{\mathrm{ad}}} j(u)$. ☐

Example (continued)**.** Let us continue with the example. Assumptions (i) and (ii) in Assumption 3.1 hold true, with $U = W^{-1,q}_0(\Omega)$. Concerning assumption (iii), we had already noted that the operator $E = \mathrm{tr}^*$ maps $L^t(\partial\Omega)$ to $W^{-1,q}_0(\Omega)$. More precisely, it does so when $t \geq q\frac{d-1}{d}$, and compactly so if the inequality is strict. Now choose $Y = L^2(\partial\Omega)$ and $X = L^p(\partial\Omega)$ with $p > q\frac{d-1}{d}$. We suppose that $p \geq 2$ is necessary. (This is always the case when $d = 3$ due to $q > d = 3$.) Then we have $E \in \mathcal{K}(L^p(\partial\Omega); W^{-1,q}_0(\Omega))$ and we prove in §4.4 that there exists $\eta > \frac{2}{2+p}$ such that $E \in \mathcal{L}((L^p(\partial\Omega), L^2(\partial\Omega))_{\eta,1}; W^{-1,q}_0(\Omega))$ whenever p is large enough. This shows that Assumption 3.1 (iii) is satisfied for the control space $\mathcal{U}_w = L^{2s}(J; W^{-1,q}_0(\Omega))$ for any $s \in [1,\infty]$. With box constraints constrained by measurable functions u_a, u_b for $\mathcal{U}_{\mathrm{ad}}$, assumption (iv) is also satisfied since clearly $\mathbb{W}^{1,2}_p(L^p(\partial\Omega), L^2(\partial\Omega)) \hookrightarrow L^2(J; L^2(\partial\Omega))$ and convergence in $L^2(J; L^2(\partial\Omega))$ preserves pointwise a.e. bounds. If we choose for instance $\mathsf{J}_s(\theta,\varphi) = \frac{1}{2}\|\theta - \theta_d\|^2_{L^2(J;L^2(\Omega))}$ with $\theta_d \in L^2(J; L^2(\Omega))$, then assumption (v) is also satisfied, and we can use Theorem 3.7 to infer that there exists an optimal control for the problem

$$
(\text{Ex-OCP}) \qquad \left\{
\begin{aligned}
\min_{(\theta,\varphi,u)} \quad & \frac{1}{2}\|\theta - \theta_d\|^2_{L^2(J;L^2(\Omega))} + \frac{\beta_2}{2}\|\partial_t u\|^2_{L^2(J;L^2(\partial\Omega))} + \frac{\beta_p}{p}\|u\|^p_{L^p(J;L^p(\partial\Omega))} \\
\text{subject to} \quad & F((\theta,\varphi),u) = 0 \qquad \text{in } \mathcal{Z}, \\
& u_a \leq u \leq u_b \qquad \text{a.e. on } J \times \partial\Omega.
\end{aligned}
\right.
$$

Here, F as in (3.2) describes the system (3.1). We require $\beta_p > 0$ if $\mathcal{U}_{\mathrm{ad}}$ is not bounded in $L^p(J; L^p(\partial\Omega))$, i.e., if $u_a, u_b \notin L^p(J; L^p(\partial\Omega))$.

We will exhibit more examples for possible situations and arrangements of spaces X, Y and appropriate choices of p for given spaces $\mathcal{U}_w$ in §4.

3.2. **Norm differentiability and duality mappings.** In the following, we will need J and thus in particular J_c to be at least Gâteaux-differentiable. Being interested in Gâteaux differentiability for J_c, it is natural to consider differentiability properties of the norm functions n_Y and n_X on Y and X.

Definition 3.8. We say that a Banach space Z is *smooth* if n_Z is Gâteaux differentiable on $Z \setminus \{0\}$. Analogously, we call Z *Fréchet smooth* if n_Z is Fréchet differentiable on $Z \setminus \{0\}$.

It is well known that a norm function on a Banach space cannot be Gáteaux differentiable in 0 since this would imply the same for $t \mapsto |t|$ as a real function. Note moreover that due to convexity, if a norm is Fréchet differentiable, then it is already continuously so, cf. [41, Cor. 4.3.4]. The following result gives us all we need for J_c:

Proposition 3.9 ([28, Thm. 3.1], [27, Thm. 2.5])**.** *Let Z be a Banach space. For $p \in (1,\infty)$, the space $L^p(J;Z)$ is (Fréchet) smooth if and only if Z is (Fréchet) smooth.*

Since J_c consists of powers of Lebesgue-Bochner norms, the foregoing proposition shows that J_c is Gâteaux differentiable on the whole space $\mathbb{W}^{1,2}_p(X;Y)$ whenever X and Y are smooth.

Smoothness is naturally linked to Banach space geometry, in particular to convexity. Recall that we say that a Banach space Z is *strictly convex* if from $\|z_1\|_Z = \|z_2\|_Z = 1$ with $z_1 \neq z_2$ it follows that $\|(1-\lambda)z_1 + \lambda z_2\|_Z < 1$ for all $\lambda \in (0,1)$.

Proposition 3.10 ([41, Prop. 4.7.10/4.7.14])**.** *Let Z be a reflexive Banach space. Then Z is smooth if and only if Z' is strictly convex and Z' is smooth if and only if Z is strictly convex.*

There exist various variations of the foregoing results and more precise assertions regarding (Fréchet) smoothness in relation to Banach space geometry. We refer to [11] and [9] for a comprehensive treatment of the topic, see also [42, Sect. 2.3]. For us it suffices to know that Hilbert spaces in general, and further the "standard" spaces of type $L^s(\Lambda)$ and $W^{m,s}(\Lambda)$ for $\Lambda \subset \mathbb{R}^n$ and $m \in \mathbb{N}$ as well as $s \in (1,\infty)$ are smooth, see for instance [9, Ex. I.3.7, Thm. II.4.7]. (There are also more "exotic" smooth spaces like Besov spaces on the real line.)

Since they will be needed often, we introduce the following mappings related to norm derivatives:

Definition 3.11 (Support and duality mapping)**.** We define the set-valued *support mapping* φ_Z of a Banach space Z to be

$$\varphi_Z(z) := \Big\{ z' \in Z' \colon \|z'\|_{Z'} \le 1,\ \langle z', z\rangle = \|z\|_Z \Big\}.$$

Further, we extend this notion to the *duality mapping* $\varphi_{Z,q}$ of order $q \in [1,\infty)$ on Z by

$$\varphi_{Z,q}(z) = \|z\|_Z^{q-1}\varphi_Z(z),$$

with of course $\varphi_{Z,1} = \varphi_Z$.

Both mappings are *a priori* set-valued in Z', and their image in each z is nonempty due to the Hahn-Banach theorem. In fact, $\varphi_X(x)$ is the subdifferential of the convex function n_X in x and indeed, if X is smooth, then φ_X is single valued on $X \setminus \{0\}$ and we have $n_X'(x) = \varphi_X(x)$ with $\|\varphi_X(x)\|_{X'} = 1$ ([41, Prop. 4.7.1]). Moreover, the derivative of n_X^p is then given by

$$(n_X^p)'(x) = p\, n_X^{p-1}(x) n_X'(x) = p\,\varphi_{X,p}(x), \tag{3.3}$$

hence $\varphi_{X,p}(x) = \frac{1}{p}(n_X^p)'(x)$. Observe also that $\varphi_{Z,q}(\alpha z) = |\alpha|^{q-2}\alpha\varphi_{Z,q}(z)$ for any $\alpha \in \mathbb{R}$; in particular, $\varphi_{Z,q}$ is an odd function.

Remark 3.12. Note that for a *Hilbert* space Z, the normalized duality mapping $\|\cdot\|_Z^{2-q}\varphi_{Z,q}$ coincides with the Riesz isometry $\Phi_Z\colon Z \to Z'$ defined by $(\bar z, z)_Z = \langle \Phi_Z(\bar z), z\rangle_{Z',Z}$ for all $z, \bar z \in Z$. In this case (and only there, [9, Prop. I.4.8]), $\|\cdot\|^{2-q}\varphi_{Z,q}$ is in fact *linear*. This is of course particularly interesting for $q = 2$.

Example 3.13. We briefly recall the support- and duality mappings for the most common spaces $L^s(\Lambda)$ and $W^{1,s}(\Lambda)$:

(i) For $X = L^s(\Lambda)$ for $s \in (1,\infty)$, the support mapping $\varphi_{L^s(\Lambda)}(u)$ is given by

$$\varphi_{L^s(\Lambda)}(u) = \frac{|u|^{s-2}u}{\|u\|_{L^s(\Lambda)}^{s-1}}$$

in accordance with the duality pairing on $L^s(\Lambda)$ as the $L^2(\Lambda)$ scalar product with the usual identifcation $(L^s(\Lambda))' = L^{s'}(\Lambda)$. This means that

$$\varphi_{L^s(\Lambda),p}(u) = \|u\|_{L^s(\Lambda)}^{p-s}|u|^{s-2}u.$$

(ii) With $\|u\|_{W^{1,s}(\Lambda)} = \left(\int_\Lambda |u|^s + |\nabla u|^s \,\mathrm{d}x\right)^{\frac{1}{s}}$, we obtain analogously

$$\varphi_{W^{1,s}(\Lambda)}(u) = \frac{|u|^{s-2}u + |\nabla u|^{s-2}\nabla u\cdot\nabla}{\|u\|_{W^{1,s}(\Lambda)}^{s-1}}$$

and

$$\varphi_{W^{1,s}(\Lambda),p}(u) = \|u\|_{W^{1,s}(\Lambda)}^{p-s}\left(|u|^{s-2}u + |\nabla u|^{s-2}\nabla u\cdot\nabla\right).$$

Another useful property of the support- and duality mappings which we will use later is the following ([9, Prop. II.3.6]):

Proposition 3.14. *Let Z be reflexive and let Z and Z' be smooth. For $r \in (1,\infty)$, the duality mapping $\varphi_{Z,r}$ maps Z bijectively to Z' with inverse $\varphi_{Z',r'}$. In particular,*

$$\varphi_{Z',r'}\circ\varphi_{Z,r} = \mathrm{id}_Z \quad \textit{and} \quad \varphi_{Z,r}\circ\varphi_{Z',r'} = \mathrm{id}_{Z'} \tag{3.4}$$

holds true.

3.3. **First order necessary conditions.** We now turn to first order necessary conditions for the reduced optimal control problem (OCP_u), i.e.,

$$\min_u \quad j(u) = \mathsf{J}(y(\mathcal{E}u), u) \qquad \text{subject to} \quad u \in \mathcal{U}_{\mathrm{ad}}. \tag{OCP_u}$$

The following additional assumptions will be needed:

Assumption 3.15. *The following differentiability properties for (OCP_u) and (OCP) are true:*

(i) *The control-to-state operator $\mathcal{U}_w \ni u \mapsto y(u) \in \mathcal{Y}$ is continuously differentiable,*
(ii) *$\mathsf{J}_s\colon \mathcal{Y} \to \mathbb{R}$ is continuously differentiable,*
(iii) *J_c is given as in (1.1), whereas Y is a Hilbert space and X is smooth, cf. §3.2,*
(iv) *$\mathcal{U}_w = L^r(J;U)$ for some $r \in [1,\infty)$ and a Banach space U such that U' has the Radon-Nikodým property.*

Classically, the implicit function theorem tells us that Assumption 3.15 (i) is satisfied if F is continuously differentiable and $\partial_y F(y,u)$ is continuously invertible as a linear operator from $\mathcal{Z}$ to $\mathcal{Y}$ for $(y,u) \in \mathcal{Y}\times\mathcal{U}_w$, i.e., the linearized evolution equation is uniquely solvable for every right-hand side in $\mathcal{Z}$ with a continuous solution operator.

Lemma 3.16. *Under Assumption 3.15, J_c is Gâteaux differentiable on $\mathbb{W}_p^{1,2}(X;Y)$ and we have*

$$\mathsf{J}_c'(u)h = \int_J \beta_2\big(\partial_t u(t), \partial_t h(t)\big)_Y + \beta_p\big\langle\varphi_{X,p}(u(t)), h(t)\big\rangle_{X',X}\,\mathrm{d}t \tag{3.5}$$

for all $u, h \in \mathbb{W}_p^{1,2}(X;Y)$.

Proof. This follows from smoothness of the Hilbert space Y, Proposition 3.9 and a chain rule for Fréchet- and Gâteaux differentiable functions together with the derivative formula for Lebesgue-Bochner norms from [27, 28]. We have also used the identification of the duality mapping $\varphi_{Y,2}$ with the Riesz isomorphism Φ_Y from Remark 3.12. □

Together with the assumptions on J_s, the reduced objective functional j is also continuously differentiable on $\mathbb{W}^{1,2}_p(X;Y)$ under Assumptions 3.1 (ii) and 3.15. For the Radon-Nikodým property, we refer to [12]. It is satisfied if U is reflexive.

Let us come to optimality conditions. The following definitions and results follow the standard theory. We begin with the notion of a locally optimal control, which is the correct concept at this point since the problem at hand is nonconvex in general.

Definition 3.17. Let $\bar u \in \mathcal{U}_{\mathrm{ad}}$, i.e., feasible for (OCP_u). We call $\bar u$ a *locally optimal control* for (OCP_u), if

$$j(\bar u) \le j(u) \quad \text{for all } u \in \mathcal{U}_{\mathrm{ad}} \text{ such that } \|u - \bar u\|_{\mathbb{W}^{1,2}_p(X;Y)} < \varepsilon$$

for some $\varepsilon > 0$.

The constraint set $\mathcal{U}_{\mathrm{ad}}$ was assumed to be closed and convex, hence local optimality for a control $\bar u \in \mathcal{U}_{\mathrm{ad}}$ implies the following variational inequality:

Theorem 3.18 (First-order optimality condition)**.** *Let $\bar u$ be a locally optimal control for* (OCP_u). *Then*

$$j'(\bar u)(u - \bar u) \ge 0 \quad \textit{for all } u \in \mathcal{U}_{\mathrm{ad}} \tag{3.6}$$

holds true.

We proceed by dissecting (3.6) by using the special structure of J_c. The aim for the rest of this chapter is to work out the benefit of having a *linear* dependence on $\partial_t \bar u$ in $j'(\bar u)$.

For the derivative $j'(u)$ in (3.6), we have by virtue of (3.5):

$$j'(u)h = \bigl\langle \mathsf{J}_s'(y_{\mathcal{E}}(u)), y'(\mathcal{E}u)\mathcal{E}h\bigr\rangle_{\mathcal{Y}',\mathcal{Y}} + \int_J \beta_2\bigl(\partial_t u(t), \partial_t h(t)\bigr)_Y + \beta_p \langle \varphi_{X,p}(u(t)), h(t)\rangle_{X',X}\ \mathrm{d}t$$

for all $h \in \mathbb{W}^{1,2}_p(X;Y)$. Hence, in the situation of Theorem 3.18, introducing the so-called *adjoint state* $\mu(\bar u) \in \mathcal{U}_w'$ with respect to $\bar u$ by $\mu(\bar u) = y'(\mathcal{E}\bar u)^* \mathsf{J}_s'(y(\bar u))$, we may rewrite (3.6) to

$$\begin{aligned} &\bigl\langle \mu(\bar u), \mathcal{E}(u - \bar u)\bigr\rangle_{\mathcal{U}_w',\mathcal{U}_w} \\ &\quad + \int_J \beta_2\bigl(\partial_t \bar u(t), \partial_t (u - \bar u)(t)\bigr)_Y + \beta_p \bigl\langle \varphi_{X,p}(\bar u(t)), u(t) - \bar u(t)\bigr\rangle_{X',X}\ \mathrm{d}t \ge 0 \end{aligned} \tag{3.7}$$

for all $u \in \mathcal{U}_{\mathrm{ad}}$. To give (3.7) a more precise meaning, we need a few additional considerations. Let us therefore recall the Riesz isomorphism $\Phi_Y \colon Y \to Y'$ on Y from Remark 3.12. For $y \in Y$, we abbreviate $\Phi_Y(y)$ by boldface $\mathbf{y}$. Given a function $w \colon J \to Y$, the distributional time derivative satisfies $\Phi_Y(\partial_t w(t)) = \partial_t \Phi_Y(w(t))$, cf. [10, Ch. XVIII, §1]. In this sense, we will just write $\partial_t \mathbf{w}(\mathbf{t})$ for these terms without abuse of notation. We obtain the following reformulation of (3.7):

Corollary 3.19. *In the situation of Theorem 3.18, we may equivalently rewrite* (3.6) *via* (3.7) *and* (3.9) *to*

$$\begin{aligned} 0 \le \int_J &\bigl\langle \partial_t \bar{\mathbf{u}}(\mathbf{t}), \partial_t (u - \bar u)(t)\bigr\rangle_{Y',Y} \\ &+ \frac{1}{\beta_2}\Bigl(\bigl\langle \beta_p \varphi_{X,p}(\bar u(t)), u(t) - \bar u(t)\bigr\rangle_{X',X} + \bigl\langle \mu_*(\bar u)(t), u(t) - \bar u(t)\bigr\rangle_{(X,Y)_{\eta,1}}\Bigr)\ \mathrm{d}t \end{aligned} \tag{3.8}$$

for all $u \in \mathcal{U}_{\mathrm{ad}}$, where $\mu_*(\bar u)(t) = E^* \mu(\bar u)(t)$, so $\mu_*(\bar u) \in L^{r'}(J; (Y', X')_{\eta,\infty})$.

Proof. By the choice $\mathcal{U}_w = L^r(J;U)$ for $r \in [1,\infty)$ with U' having the Radon-Nikodým property as in Assumption 3.15, we have $\mathcal{U}_w' = L^{r'}(J;U')$ ([12, Thm. IV.1.1]) and

$$\langle \xi, f\rangle_{\mathcal{U}_w',\mathcal{U}_w} = \int_J \big\langle \xi(t), f(t)\big\rangle_{U',U} \,\mathrm{d}t \quad \text{for} \quad \xi \in \mathcal{U}_w' \text{ and } f \in \mathcal{U}_w$$

Recall that by assumption $E \in \mathcal{L}((X,Y)_{\eta,1};U)$ and hence $E^* \in \mathcal{L}(U';(Y',X')_{\eta,\infty})$, where $(Y',X')_{\eta,\infty} = (X,Y)_{\eta,1}'$, cf. [46, Ch. 1.11.2/1.11.3]. Hence we obtain that

$$\langle \mu(\bar u), \mathcal{E}h\rangle_{\mathcal{U}_w',\mathcal{U}_w} = \int_J \big\langle \mu(\bar u)(t), Eh(t)\big\rangle_{U',U} \,\mathrm{d}t = \int_J \big\langle E^*\mu(\bar u)(t), h(t)\big\rangle_{(X,Y)_{\eta,1}} \,\mathrm{d}t \tag{3.9}$$

for every $h \in \mathbb{W}^{1,2}_p(X;Y)$. This is meaningful since $h \in \mathbb{W}^{1,2}_p(X;Y) \hookrightarrow L^s(J;(X,Y)_{\eta,1})$ for $\frac{1}{s} > \frac{1-\eta}{p} - \frac{\eta}{2}$, see Remark 2.10, so we can choose $s \geq r$ and the integrabilities in (3.9) match. □

We now proceed with further reformulations of (3.8). Of course, the interesting question is whether $\bar u$ is from $\operatorname{int}(\mathcal{U}_{\mathrm{ad}})$ or not. The first case is to be interpreted as the one where $\mathcal{U}_{\mathrm{ad}}$ is the whole space $\mathbb{W}^{1,2}_p(X;Y)$, i.e., there are in fact no control constraints present in (OCP_u), since we are in general unable to determine *a priori* whether $\bar u \in \operatorname{int}(\mathcal{U}_{\mathrm{ad}})$ or not.

3.3.1. *No control constraints.* If $\mathcal{U}_{\mathrm{ad}} = \mathbb{W}^{1,2}_p(X;Y)$, then the variational inequality (3.8) is in fact an equality, namely

$$\int_J \beta_2\big\langle \partial_t \bar{\mathbf{u}}(\mathbf{t}), \partial_t h(t)\big\rangle_{Y',Y} + \big\langle \beta_p \varphi_{X,p}(\bar u(t)), h(t)\big\rangle_{X',X} + \big\langle \mu_*(\bar u)(t), h(t)\big\rangle_{(X,Y)_{\eta,1}} \,\mathrm{d}t = 0 \tag{3.10}$$

for all $h \in \mathbb{W}^{1,2}_p(X;Y)$. The aim is now to rewrite the foregoing equality to an "ordinary" differential equation in a Banach space using an integration by parts formula as in Theorem A.1 in the appendix.

Lemma 3.20. *Let the assumptions of Corollary 3.19 be given and assume that* $\mathcal{U}_{\mathrm{ad}} = \mathbb{W}^{1,2}_p(X;Y)$. *Then* (3.6) *is equivalent to* $\bar u \in \mathbb{W}^{1,2}_p(X;Y)$ *being a solution to the abstract differential equation*

$$\beta_2 \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) = \beta_p \varphi_{X,p}(\bar u(t)) + \mu_*(\bar u)(t) \quad \text{in } X', \tag{3.11}$$

for almost all $t \in J$, *with the boundary conditions* $\partial_t \bar{\mathbf{u}}(\mathbf{T_0}) = \partial_t \bar{\mathbf{u}}(\mathbf{T_1}) = 0$. *In particular, we have* $\partial_t^2 \bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$.

Proof. From Corollary 3.19, we obtain that (3.6) is equivalent to (3.10) in the case $\bar u \in \operatorname{int}(\mathcal{U}_{\mathrm{ad}})$. Now let us choose $h \in C_c^\infty(J) \otimes X \subset \mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,2}_r((X,Y)_{\eta,1},Y)$ in the form $h = \phi \otimes f$ with $\phi \in C_c^\infty(J)$ and $f \in X$. Then we have ([10, Ch. XVIII, §1])

$$\Big\langle \int_J \phi'(t)\, \beta_2 \partial_t \bar{\mathbf{u}}(\mathbf{t}) \,\mathrm{d}t, f\Big\rangle_{Y',Y} = -\Big\langle \int_J \phi(t)\big(\beta_p\varphi_{X,p}(\bar u(t)) + \mu_*(\bar u)(t)\big) \,\mathrm{d}t, f\Big\rangle_{X',X}$$

for all $f \in X$, i.e.,

$$\int_J \phi'(t)\, \beta_2 \partial_t \bar{\mathbf{u}}(\mathbf{t}) \,\mathrm{d}t = -\int_J \phi(t)\big(\beta_p\varphi_{X,p}(\bar u(t)) + \mu_*(\bar u)(t)\big) \,\mathrm{d}t \quad \text{in } X'$$

and this is true for all $\phi \in C_c^\infty(J)$. (Recall that $(Y',X')_{\eta,\infty} \hookrightarrow X'$, so $\mu_*(\bar u)(t)$ can be seen as an element of X'.) But this means exactly that

$$\beta_2 \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) = \beta_p \varphi_{X,p}(\bar u(t)) + \mu_*(\bar u)(t) \quad \text{in } X' \tag{3.12}$$

in the distributional sense for almost all $t \in J$. Hence, $\partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$. Since the latter space is a subset of $L^{\min(p',r')}(J;X')$, we obtain that $\partial_t^2 \bar{\mathbf{u}}$ is in fact a weak derivative. Further, we are now able to apply Theorem A.1: Let $h \in \mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,2}_r((X,Y)_{\eta,1};Y)$

be arbitrary. Then we have $\partial_t \bar{\mathbf{u}} \in L^2(J;Y')$ and $\partial_t^2 \bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ as well as $h \in L^p(J;X) \cap L^r(J;(X,Y)_{\eta,1})$ and $\partial_t h \in L^2(J;Y)$, so

$$\begin{aligned}(3.13)\quad &\int_J \langle \partial_t \bar{\mathbf{u}}(\mathbf{t}), \partial_t h(t)\rangle_{Y',Y} \\ &= -\int_J \langle \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}), h(t)\rangle_{X',X}\,\mathrm{d}t + \langle h(T_1), \partial_t \bar{\mathbf{u}}(\mathbf{T_1})\rangle_\xi - \langle h(T_0), \partial_t \mathbf{u}(\mathbf{T_0})\rangle_\xi,\end{aligned}$$

is true for all $h \in \mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,2}_r((X,Y)_{\eta,1};Y)$, where $\langle\cdot,\cdot\rangle_\xi$ stands for the duality pairing of

$$X_\xi = (X,Y)_{\frac{2}{2+p},\frac{2+p}{2}} \cap (X,Y)_{\frac{2+\eta r}{2+r},\frac{2+r}{2}}$$

with its dual space

$$X_\xi' = (Y',X')_{\frac{2}{2+p},\frac{2+p}{p}} + (Y',X')_{\frac{2+\eta r}{2+r},\frac{2+r}{r}},$$

cf. Lemma 2.7. We have used the reiteration theorem to obtain

$$\big((X,Y)_{\eta,1},Y\big)_{\frac{2}{2+r},\frac{2+r}{2}} = (X,Y)_{\frac{2+\eta r}{2+r},\frac{2+r}{2}}$$

and analogously for the dual space. Finally, inserting (3.10) and (3.12) into (3.13), we find that $\partial_t \bar{\mathbf{u}}(\mathbf{T_1}) = \partial_t \bar{\mathbf{u}}(\mathbf{T_0}) = 0$ in X_ξ', since $h \in \mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,2}_r((X,Y)_{\eta,1},Y)$ was arbitrary.

For the reverse implication, we test (3.12) with $h = \phi \otimes f \in C_c^\infty(\overline{J}) \otimes X$ and repeat the foregoing actions in reverse. Since $C_c^\infty(\overline{J}) \otimes X$ is dense in $\mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,r}_r((X,Y)_{\eta,1}),Y)$ by [2, Thm. V.2.4.6], this implies that (3.10) holds true for all $h \in \mathbb{W}^{1,2}_p(X;Y) \cap \mathbb{W}^{1,2}_r((X,Y)_{\eta,1};Y)$. □

Example (continued)**.** We look at Lemma 3.20 in the context of the running example. We had $Y = L^2(\partial\Omega)$ and $X = L^p(\partial\Omega)$ with $p > q\frac{d-1}{d}$. Accordingly, $\varphi_{X,p}(\bar{u}(t))$ is given by $|\bar{u}(t)|^{p-2}\bar{u}(t)$. We had $\mathcal{U}_w = L^{2s}(J;W_0^{-1,q}(\Omega))$ and set $r := 2s$. The adjoint state $\mu(\bar{u}) \in \mathcal{U}_w' = L^{r'}(J;W^{1,q'}(\Omega))$ is given by the component ψ of the (very weak) solution (ϑ,ψ) to the adjoint system

$$(3.14)\qquad \left\{\begin{aligned} \partial_t\vartheta - \Delta\vartheta &= \theta(\bar{u}) - \theta_d && \text{in } \Omega, && \nabla\vartheta\cdot\nu = 0 \ \text{ on } \partial\Omega, && \vartheta(T) = 0,\\ -\Delta\psi &= -2\operatorname{div}(\vartheta\nabla\varphi(\bar{u})) && \text{in } \Omega, && \nabla\psi\cdot\nu = 0 \ \text{ on } \partial\Omega, \end{aligned}\right.$$

and $\mu_*(\bar{u}) \in L^{r'}(J;(L^2(\partial\Omega),L^{p'}(\partial\Omega))_{\eta,\infty})$ is its spatial boundary trace $\operatorname{tr}\psi$. (Here it becomes visible that $\mu(\bar{u})$ and $\mu_*(\bar{u})$ in general depend *nonlocally in time* upon $\bar{u}$.) This means that by Lemma 3.20, a locally optimal control $\bar{u} \in \mathbb{W}^{1,2}_p(L^p(\partial\Omega),L^2(\partial\Omega))$ for (Ex-OCP) satisfies (3.11), so

$$\beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) = \beta_p|\bar{u}(t)|^{p-2}\bar{u}(t) + \operatorname{tr}\psi(t) \quad \text{in } L^{p'}(\partial\Omega)$$

for almost all $t \in J$, with $\partial_t\bar{\mathbf{u}}(\mathbf{T_0}) = \partial_t\bar{\mathbf{u}}(\mathbf{T_1}) = 0$ almost everywhere on $\partial\Omega$ and the regularity $\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) \in L^{p'}(J;L^{p'}(\partial\Omega)) + L^{r'}(J;(L^2(\partial\Omega),L^{p'}(\partial\Omega))_{\eta,\infty})$. Note that since $2 \geq p'$, the Lorentz space $(L^2(\partial\Omega),L^{p'}(\partial\Omega))_{\eta,\infty}$ embeds into $L^{\tau'}(\partial\Omega)$ for $\frac{1}{\tau'} > \frac{1-\eta}{p'} + \frac{\eta}{2}$, see [46, Thm. 1.18.6.2].

3.3.2. *The general case of control constraints.* Let us return to (3.8) and consider the general case where $\bar{u}$ is *not* an interior point of $\mathcal{U}_{\mathrm{ad}}$. For this case, we have to treat the variational inequality directly. The aim is to characterize $\bar{u}$ using projections in X and X', respectively. Such a characterization could then be used in numerical methods such as semismooth Newton methods. See also Remark 3.31 below.

Recall that we have assumed X to be smooth. We will also need that X' is smooth, or, equivalently, that X is strictly convex, see Proposition 3.10. The following assumption is thus supposed to be valid from now on.

Assumption 3.21. *The space X' is smooth. Equivalently, X is strictly convex.*

We first consider the usual *metric* projection:

Definition 3.22. The metric projection $P_{K,\varsigma}(y) \in K$ of a point $y \in X$ onto the nonempty closed convex set $K \subseteq X$ with parameter $\varsigma \in (1,\infty)$ is given by

$$\bar{x} = P_{K,\varsigma}(y) \quad\Longleftrightarrow\quad \bar{x} = \operatorname*{arg\,min}_{z\in K} \frac{1}{\varsigma}\|z-y\|_X^\varsigma. \tag{3.15}$$

We write P_K instead of $P_{K,2}$.

It is clear that the differentiable convex minimization problem in (3.15) has a solution in K and this solution is unique since $\|\cdot\|_X^\varsigma$ is a strictly convex function for $\varsigma > 1$. (Here we have used that X is strictly convex and [9, Prop. II.1.6].)

Lemma 3.23. *Let $K \subseteq X$ be a nonempty closed and convex set, let $\bar{x} \in K$ and let $f\colon X \to X'$. Then*

$$\big\langle f(\bar{x}), x-\bar{x}\big\rangle_{X',X} \geq 0 \quad \textit{for all } x \in K \qquad\Longleftrightarrow\qquad \bar{x} = P_{K,\varsigma}\big(\bar{x} - \varphi_{X',\varsigma'}(\alpha f(\bar{x}))\big) \tag{3.16}$$

for every $\varsigma > 1$ and $\alpha > 0$, where $\varphi_{X',\varsigma'}$ is the duality mapping as in Definition 3.11.

Proof. This follows from the observation that the identity $\bar{x} = P_{K,\varsigma}(y)$ for $y \in K$ is equivalent to optimality condition for (3.15), so

$$\big\langle \varphi_{X,\varsigma}(\bar{x}-y), x-\bar{x}\big\rangle_{X',X} \geq 0 \quad \text{for all } x \in K. \tag{3.17}$$

(Recall from (3.3) that $\varphi_{X,\varsigma}(\bar{x}) = \frac{1}{\varsigma}(n_X^\varsigma)'(\bar{x})$.) For $y = \bar{x} - \varphi_{X',\varsigma'}(\alpha f(\bar{x}))$, the foregoing inequality collapses to the variational inequality in (3.16). Here, we have used (3.4) from Proposition 3.14 which also requires X' to be smooth. □

In the case $\varsigma = 2$ and X (and thus X') being a Hilbert space, the formula for $\bar{x}$ in (3.16) becomes exactly $\bar{x} = P_K(\bar{x} - \alpha\Phi_X^{-1}(f(\bar{x})))$, recall Remark 3.12.

In order to introduce a *generalized* projection on X, we follow [1, 29] and define the mapping

$$V_\varsigma\colon X' \times X \to \mathbb{R}, \qquad V_\varsigma(\phi, x) := \frac{1}{\varsigma'}\|\phi\|_{X'}^{\varsigma'} - \langle \phi, x\rangle_{X',X} + \frac{1}{\varsigma}\|x\|_X^\varsigma$$

for $\varsigma \in (1,\infty)$. The next definition originates from [1].

Definition 3.24. The generalized projection $\pi_{K,\varsigma}(\phi) \in K$ of a functional $\phi \in X'$ onto the nonempty closed and convex set $K \subseteq X$ with parameter $\varsigma \in (1,\infty)$ is given by

$$\bar{x} = \pi_{K,\varsigma}(\phi) \quad\Longleftrightarrow\quad \bar{x} = \operatorname*{arg\,min}_{x\in K} V_\varsigma(\phi, x). \tag{3.18}$$

We write π_K instead of $\pi_{K,2}$.

One readily observes that $V_\varsigma(\phi,x) \geq 0$ for all $(\phi,x) \in X' \times X$, such that the minimum in (3.18) is indeed a finite value. For the proof that there in fact exists a *minimum* in (3.18) instead of a mere infimum, uniqueness of that minimum, and more properties of $\pi_{K,2}$ in general, we refer to [29], where the obvious modifications have to be applied to account for $\varsigma \neq 2$. Of course, uniqueness again relies on the strict convexity of X.

Note that if X is a Hilbert space, then $V_2(\phi,x) = \frac{1}{2}\|\Phi_X^{-1}(\phi) - x\|_X^2$ and $\pi_K \circ \Phi_X$ coincides with the metric projection P_K. In this sense, Definition 3.24 indeed generalizes the notion of a projection. The following property for the generalized projection which we need in the following is in analogy to Lemma 3.23 above:

Lemma 3.25. *Let $K \subseteq X$ be a nonempty closed and convex set, let $\bar{x} \in K$ and let $f\colon X \to X'$. Then*

$$\big\langle f(\bar{x}), x-\bar{x}\big\rangle_{X',X} \geq 0 \quad \textit{for all } x \in K \qquad\Longleftrightarrow\qquad \bar{x} = \pi_{K,\varsigma}\big(\varphi_{X,\varsigma}(\bar{x}) - \alpha f(\bar{x})\big), \tag{3.19}$$

for every $\varsigma \in (1,\infty)$ and $\alpha > 0$, where $\varphi_{X,\varsigma}$ is the duality mapping as in Definition 3.11.

Proof. The proof is analogous to that of Lemma 3.23: We know that $\bar{x}$ must satisfy the optimality conditions of the minimization problem in (3.18), so

$$\bar{x} = \pi_{K,\varsigma}(\phi) \qquad \Longleftrightarrow \qquad \big\langle \partial_x V_\varsigma(\phi,\bar{x}), x - \bar{x}\big\rangle_{X',X} \geq 0 \quad \text{for all } x \in K,$$

and we have $\partial_x V_\varsigma(\phi,\bar{x}) = \varphi_{X,\varsigma}(\bar{x}) - \phi$. Hence, inserting $\phi = \varphi_{X,\varsigma}(\bar{x}) - \alpha f(\bar{x})$ gives the claim. □

Remark 3.26. Let us, analogously to Example 3.13, calculate an expression for the generalized projection in $X = L^s(\Lambda)$ for some $\Lambda \subset \mathbb{R}^n$ and $s \in (1,\infty)$, where the convex closed set K is given by so-called box-constraints:

$$K = \Big\{ u \in L^s(\Lambda) \colon u_{\min}(\mathrm{x}) \leq u(\mathrm{x}) \leq u_{\max}(\mathrm{x}) \text{ f.a.a. } \mathrm{x} \in \Lambda \Big\}. \tag{3.20}$$

Here, $u_{\min}, u_{\max}$ are measurable functions which satisfy $u_{\min} \leq u_{\max}$ a.e. in Λ and we assume $K \neq \emptyset$. It is well-known—and, using Example 3.13, easy to see—that in this case the metric projection $P_{K,\varsigma}$ in $L^s(\Lambda)$ for a function $\xi \in L^s(\Lambda)$ and $\varsigma \in (1,\infty)$ is given by

$$\psi = P_{K,\varsigma}(\xi) \quad \text{in } L^s(\Lambda) \qquad \Longleftrightarrow \qquad \psi(\mathrm{x}) = \mathbb{P}_{[u_{\min}(\mathrm{x}),u_{\max}(\mathrm{x})]}\big(\xi(\mathrm{x})\big) \quad \text{f.a.e. } \mathrm{x} \in \Lambda, \tag{3.21}$$

where $\mathbb{P}_{[a,b]}$ denotes the usual pointwise projection onto the interval $[a,b]$ in $\mathbb{R}$ for given $a, b \in \mathbb{R}$ with $a \leq b$, i.e., $\mathbb{P}_{[a,b]}(\mathrm{y}) = \min(b, \max(\mathrm{y}, a))$. We will see that we obtain an analogous formula for the generalized projection $\pi_{K,s}$, thereby justifying the expression *generalized projection* also on a more practical level, cf. (3.22) below. Proposition 3.14 will become useful now. Indeed, the $L^s(\Lambda)$ spaces are known to be strictly convex for $s \in (1,\infty)$. Thus, the duality mappings $\varphi_{L^s(\Lambda),\varsigma}$ and $\varphi_{L^{s'}(\Omega),\varsigma'}$ transform $L^s(\Lambda)$ bijectively into $L^{s'}(\Omega)$ and vice versa, and are inverse to each other, for every $\varsigma \in (1,\infty)$.

Now let us consider $\zeta \in L^{s'}(\Lambda)$. By Lemma 3.25 with $\alpha = 1$, we have for every $\varsigma \in (1,\infty)$

$$\phi = \pi_{K,\varsigma}(\zeta) \quad \text{in } L^s(\Lambda) \qquad \Longleftrightarrow \qquad \big\langle \varphi_{L^s(\Lambda),\varsigma}(\phi) - \zeta, y - \phi \big\rangle_{L^{s'}(\Lambda),L^s(\Lambda)} \geq 0 \quad \text{for all } y \in K.$$

Choosing $\varsigma = s$, we obtain (see Example 3.13 for the derivative formula)

$$\big\langle \varphi_{L^s(\Lambda),s}(\phi) - \zeta, y - \phi \big\rangle_{L^{s'}(\Lambda),L^s(\Lambda)} = \int_\Lambda \big(|\phi|^{s-2}\phi - \zeta\big) \cdot (y - \phi)\,\mathrm{dx}.$$

From here, one readily observes that, due to monotonicity of the duality mapping ([9, Thm. II.1.8]),

$$\begin{aligned} &\phi = \pi_{K,s}(\zeta) \quad \text{in } L^s(\Lambda) \\ &\qquad \Longleftrightarrow \qquad \phi(\mathrm{x}) = \mathbb{P}_{[u_{\min}(\mathrm{x}),u_{\max}(\mathrm{x})]}\Big(\big(\varphi_{L^{s'}(\Lambda),s'}(\zeta)\big)(\mathrm{x})\Big) \quad \text{f.a.a. } \mathrm{x} \in \Lambda. \end{aligned} \tag{3.22}$$

This means that $\pi_{K,s}$ indeed also acts from $L^{s'}(\Lambda)$ to $L^s(\Lambda)$ as the pointwise projection onto the admissible set K, in this case however necessarily combined with the duality mapping which shifts ζ from $L^{s'}(\Lambda)$ into the space $L^s(\Lambda)$. In fact, for the $L^s(\Lambda)$ spaces and box constraints we obtain $P_{K,s} = \pi_{K,s} \circ \varphi_{L^s(\Lambda),s}$ or $\pi_{K,s} = P_{K,s} \circ \varphi_{L^{s'}(\Lambda),s'}$, respectively.

To use the generalized projection to obtain a pointwise representation for $\bar{u}(t)$, we have to get rid of the integration in the variational inequality (3.8). To do so, we pose the following additional assumptions:

Assumption 3.27. *We assume the following to be true:*

(i) *the optimal control* $\bar{u}$ *has the additional regularity*

$$\partial_t^2 \bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty}) \quad \textit{with} \quad \partial_t \bar{\mathbf{u}}(\mathbf{T_1}) = \partial_t \bar{\mathbf{u}}(\mathbf{T_0}) = 0,$$

(ii) *the feasible set* $\mathcal{U}_{\mathrm{ad}}$ *is given by*

$$\mathcal{U}_{\mathrm{ad}} = \Big\{ u \in \mathbb{W}^{1,2}_p(X;Y) \colon u(t) \in U_{\mathrm{ad}} \textit{ for a.e. } t \in J \Big\} \tag{3.23}$$

for a closed convex set $U_{\mathrm{ad}} \subseteq X$.

Let us point out that the first of the foregoing assumptions is of rather delicate nature, and we consider it in more detail in §3.4.

Remark 3.28. A particular case for the assumption on the set $\mathcal{U}_{\rm ad}$ in Assumption 3.27 to be satisfied is the one of time-invariant box-constraints, cf. also Remark 3.26:

$$U_{\rm ad} = \Big\{ v \in U : u_{\min}(\mathrm{x}) \le v(\mathrm{x}) \le u_{\max}(\mathrm{x}) \text{ f.a.a. } \mathrm{x} \in \Lambda \Big\}$$

with the bounds $u_{\min}, u_{\max} \in X$.

Using these new assumptions, we obtain the following reformulation of the KKT condition (3.6):

Lemma 3.29. *Suppose Assumption 3.27. Then* (3.6) *holds true if and only if*

$$\bar{u}(t) = \pi_{U_{\rm ad},\varsigma}\left(\alpha \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) - \left(\frac{\alpha\beta_p}{\beta_2} \varphi_{X,p}(\bar{u}(t)) - \varphi_{X,\varsigma}(\bar{u}(t)) \right) - \frac{\alpha}{\beta_2} \mu_*(\bar{u})(t) \right) \quad \text{in } X \tag{3.24}$$

for $\varsigma \in (1,\infty)$ *and* $\alpha > 0$*, and almost every* $t \in J$*.*

Note that for $\varsigma = p$ and $\alpha\beta_p = \beta_2$, the duality mappings in (3.24) cancel each other.

Proof of Lemma 3.29. The KKT condition (3.6) is equivalent to (3.8) via Corollary 3.19. Now, the regularity of $\partial_t^2 \bar{\mathbf{u}}(\mathbf{t})$ together with its boundary values allows to use Theorem A.1 to obtain

$$\int_J \big\langle \partial_t \bar{\mathbf{u}}(\mathbf{t}), \partial_t (u - \bar{u})(t) \big\rangle_{Y',Y} \, \mathrm{d}t = - \int_J \big\langle \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}), u(t) - \bar{u}(t) \big\rangle_{X',X} \, \mathrm{d}t,$$

i.e., (3.8) holds true if and only if the inequality

$$\int_J \big\langle -\beta_2 \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) + \beta_p \varphi_{X,p}(\bar{u}(t)) + \mu_*(\bar{u})(t), u(t) - \bar{u}(t) \big\rangle_{X',X} \, \mathrm{d}t \ge 0 \tag{3.25}$$

does so for all $u \in \mathcal{U}_{\rm ad}$. Clearly, the left-hand side of (3.25) is continuous w.r.t. u in the $L^p(J;X) \cap L^r(J;(X,Y)_{\eta,1})$-topology, and $\mathbb{W}_p^{1,2}(X;Y)$ is dense in that space since $C_c^\infty(\overline{J};X)$ is. This means that (3.25) is true for all $u \in \mathcal{U}_{\rm ad}$ if and only if it is true for all $u \in \mathcal{U}_{\rm ad}^* := \{ w \in L^p(J;X) \cap L^r(J;(X,Y)_{\eta,1} : w(t) \in U_{\rm ad}$ for a.e. $t \in J\}$. Now Assumption 3.27 (ii) allows to deduce that for every $t \in J$ and every $v \in U_{\rm ad}$, there exists $u \in \mathcal{U}_{\rm ad}^*$ such that $u(t) = v$ holds true–namely, the constant function $u(\cdot) \equiv v$. Hence, (3.25) holds true for all $u \in \mathcal{U}_{\rm ad}$ if and only if

$$\left\langle -\partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) + \frac{\beta_p}{\beta_2} \varphi_{X,p}(\bar{u}(t)) + \frac{1}{\beta_2} \mu_*(\bar{u})(t), v - \bar{u}(t) \right\rangle_{X',X} \ge 0 \quad \text{for all } v \in U_{\rm ad}$$

for almost every $t \in J$. Now an application of Lemma 3.25 for almost every $t \in J$ yields the claim. ☐

Considering equation (3.16) we get the analogue to Lemma 3.29 for the metric projection:

Lemma 3.30. *Suppose Assumption 3.27. Then* (3.6) *holds true if and only if*

$$\bar{u}(t) = P_{U_{\rm ad},\varsigma}\left(\bar{u}(t) - \varphi_{X',\varsigma'}\Big(\frac{\alpha\beta_p}{\beta_2} \varphi_{X,p}(\bar{u}(t)) - \alpha \partial_t^2 \bar{\mathbf{u}}(\mathbf{t}) + \frac{\alpha}{\beta_2} \mu_*(\bar{u})(t) \Big) \right) \quad \text{in } X \tag{3.26}$$

for $\varsigma \in (1,\infty)$ *and* $\alpha > 0$*, and almost every* $t \in J$*.*

We have chosen to present the generalized projection since formula (3.24) looks and seems much more familiar to the Hilbert space case than (3.26). However, both formulas are equivalent and in that sense of equal value. They should serve as a possible starting point for numerical methods used to determine solutions to the optimal control problem.

Remark 3.31. When aiming to prove convergence for semismooth Newton methods in the particular case $X = L^s(\Lambda)$, one way is to interpret (3.8) as the optimality conditions for another optimization problem over $\mathbb{W}^{1,2}_p(X;Y)$ with the admissible set $\mathcal{U}_{\mathrm{ad}}$ and derive the corresponding dual problem. This leads to an optimization problem which is posed in $L^{r'}(J;L^{s'}(\Lambda))$ for which the classical semismooth Newton method is available. We refer to [47] for a comprehensive treatment. Let us also note that the proof of convergence for semismooth Newton methods relies on a regularity gap (compactness). For the classical obstacle problem, this is obtained by the additional $W^{2,q}(\Lambda)$-regularity for the solution in the $H^1_0(\Lambda)$-setting, cf. [47, Ch. 9.2]. In our setup, the additional regularity $\partial_t^2\bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ with $\partial_t\bar{\mathbf{u}}(\mathbf{T_1}) = \partial_t\bar{\mathbf{u}}(\mathbf{T_0}) = 0$ as in Lemma 3.29 or Assumption 3.27 plays this role. We leave the details for now and plan to investigate this matter further in later works.

Example (continued)**.** We consider the results in this subsection with respect to the example (Ex-OCP), for which $X = L^p(\partial\Omega)$ with $p \geq 2$ and $Y = L^2(\partial\Omega)$. It was already mentioned that $X' = L^{p'}(\partial\Omega)$ is smooth. Let the box constraints bounds u_a, u_b be given by $u_a(t) = u_{\max}$ and $u_b(t) = u_{\min}$ for almost every $t \in J$ with fixed functions $u_{\min}, u_{\max} \in L^p(\partial\Omega)$. Then Assumption 3.27(ii) is satisfied, cf. Remark 3.28. For the moment, we suppose that Assumption 3.27(i) is satisfied. We will verify that this is the case in Section 3.4 below. As seen in the example considerations in Section 3.3.1, the adjoint state $\mu(\bar{u}) \in \mathcal{U}'_w = L^{r'}(J;W^{1,q'}(\Omega))$ is given by the component ψ of the (very weak) solution (ϑ,ψ) to the adjoint system (3.14) and $\mu_*(\bar{u}) \in L^{r'}(J;(L^2(\partial\Omega),L^{p'}(\partial\Omega))_{\eta,\infty})$ is its spatial boundary trace $\operatorname{tr}\psi$.

Since $u_a, u_b \in L^\infty(J;L^p(\partial\Omega))$, we are allowed to have $\beta_p = 0$. Then $\alpha = \beta_2$ and $\varsigma = p$ in Lemma 3.29 yield that a locally optimal control $\bar{u}$ to the example optimal control problem satisfies

$$\bar{u}(t) = \pi_{U_{\mathrm{ad}},p}\Big(\beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) + |\bar{u}(t)|^{p-2}\bar{u}(t) - \operatorname{tr}\psi(t)\Big) \quad \text{in } L^p(\partial\Omega)$$

for almost every $t \in J$. In Remark 3.26 it was already observed that the generalized projection onto the box constraints in a Lebesgue space can then be resolved in a pointwise manner to obtain

$$\bar{u}(t)(\mathrm{x}) = \mathbb{P}_{[u_{\min}(\mathrm{x}),u_{\max}(\mathrm{x})]}\Big(\varphi_{L^{p'}(\partial\Omega),p'}\Big(\beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) + |\bar{u}(t)|^{p-2}\bar{u}(t) - \operatorname{tr}\psi(t)\Big)(\mathrm{x})\Big) \tag{3.27}$$

for almost all $(t,\mathrm{x}) \in J \times \partial\Omega$.

On the other hand, for $\beta_p > 0$, we can take $\alpha = \frac{\beta_2}{\beta_p}$ to obtain

$$\bar{u}(t) = \pi_{U_{\mathrm{ad}},p}\Big(\frac{\beta_2}{\beta_p}\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) - \frac{1}{\beta_p}\operatorname{tr}\psi(t)\Big) \quad \text{in } L^p(\partial\Omega).$$

for almost every $t \in J$, and analogously to the above

$$\bar{u}(t)(\mathrm{x}) = \mathbb{P}_{[u_{\min}(\mathrm{x}),u_{\max}(\mathrm{x})]}\Big(\beta_p^{1-p}\,\varphi_{L^{p'}(\partial\Omega),p'}\Big(\beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) - \operatorname{tr}\psi(t)\Big)(\mathrm{x})\Big) \tag{3.28}$$

for almost all $(t,\mathrm{x}) \in J \times \partial\Omega$.

Let us now turn to the metric projection. From Lemma 3.30 we get with $\varsigma = p$ and $\alpha = \beta_2$ for a locally optimal control

$$\bar{u}(t) = P_{U_{\mathrm{ad}},p}\Big(\bar{u}(t) - \varphi_{L^{p'}(\partial\Omega),p'}\Big(\beta_p|\bar{u}(t)|^{p-2}\bar{u}(t) - \beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) + \operatorname{tr}\psi(t)\Big)\Big) \quad \text{in } L^p(\partial\Omega)$$

for almost every $t \in J$. Hence,

$$\bar{u}(t)(\mathrm{x}) = \mathbb{P}_{[u_{\min}(\mathrm{x}),u_{\max}(\mathrm{x})]}\Big(\bar{u}(t)(\mathrm{x}) - \varphi_{L^{p'}(\partial\Omega),p'}\Big(\beta_p|\bar{u}(t)|^{p-2}\bar{u}(t) - \beta_2\partial_t^2\bar{\mathbf{u}}(\mathbf{t}) + \operatorname{tr}\psi(t)\Big)(\mathrm{x})\Big)$$

for almost all $(t,\mathrm{x}) \in J \times \partial\Omega$. The foregoing equation together with (3.27) and (3.28) give a full pointwise representation of an optimal control $\bar{u}$ for (Ex-OCP). Note that the duality mappings on Lebesgue spaces $\varphi_{L^s(\Lambda),s}(u) = |u|^{s-2}u$ are in general only $\min(1,s-1)$-Hölder continuous [42, Thm. 2.42/Ex. 2.47]. This is to account for the infinite slope in zero if $s \in (1,2)$.

3.4. **Verification of Assumption 3.27(i) in special cases.** It was already mentioned that Assumption 3.27(i) is a sensible one. This section shows why that is the case and how one may verify it in specific situations. For brevity, we fix from now on

$$g(t) := -\frac{\beta_p}{\beta_2}\varphi_{X,p}(\bar u) - \frac{1}{\beta}\mu_*(\bar u) \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty}).$$

The idea is as follows: We interpret the variational inequality (3.8) as the optimality condition for the auxiliary optimization problem

$$\min_{u\in\mathcal{U}_{\mathrm{ad}}} \int_J \frac{1}{2}\big\|\partial_t u(t)\big\|_Y^2 - \big\langle g(t), u(t)\big\rangle_{X',X}\,\mathrm{d}t.$$

With the aim of deriving additional regularity for $\partial_t^2\bar{\mathbf{u}}(\mathbf{t})$ in mind, this suggests to consider regularized versions of that problem whose optimal solutions we expect to exhibit that regularity. The regularized problems are, parametrized by the regularization parameter $\lambda > 0$,

$$(\mathrm{P}_\lambda) \qquad \min_{u\in\mathbb{W}_p^{1,2}(X;Y)} F_\lambda(u)$$

with

$$F_\lambda(u) := \int_J \frac{1}{2}\big\|\partial_t u(t)\big\|_Y^2 - \big\langle g(t), u(t)\big\rangle_{X',X} + \frac{1}{p}\left(\big\|u(t)-\bar u(t)\big\|_X^p + \frac{1}{\lambda}\big\|u(t) - P_X(u(t))\big\|_X^p\right)\mathrm{d}t.$$

We have used (as we will do from now on) P_X as a shortcut for $P_{U_{\mathrm{ad}}}$ in X.

Due to coercivity of F_λ on $\mathbb{W}_p^{1,2}(X;Y)$ and strict convexity of (powers of norms of) the spaces in F_λ, the regularized problems (P_λ) admit unique solutions $u_\lambda \in \mathbb{W}_p^{1,2}(X;Y)$. These are then shown to converge to $\bar u$ in $\mathbb{W}_p^{1,2}(X;Y)$, and we will see that if we manage to bound $\partial_t^2\mathbf{u}_\lambda$ uniformly in $L^{p'}(J;X')$, then we will also obtain the desired $\partial_t^2\bar{\mathbf{u}} \in L^{p'}(J;X')$. It will however turn out that the latter requires quite strong assumptions on (the metric projections on) the underlying spaces X and Y which *essentially* limits the treatment to the case $X = L^s(\Lambda)$ and $Y = L^2(\Lambda)$. See Lemma 3.37 and Lemma 3.39 below. The $L^{r'}(J;(Y',X')_{\eta,\infty})$ regularity comes from g only and is thus uniform in λ.

To start with, we pose some compatibility assumptions on the metric projections in X and Y.

Assumption 3.32. *For the rest of this section, we assume $p \geq 2$ and the following:*

(i) *The projection P_Y onto U_{ad} in Y continuously maps X to X and there are constants $m, b \in \mathbb{R}$ such that the following sublinear growth condition is satisfied:*

$$\|P_Y(u)\|_X \leq m\|u\|_X + b.$$

(ii) *The pointwise feasible set U_{ad} is such that P_Y is directionally differentiable on Y.*

While the Assumption 3.32(i) is not severely restrictive, some words concerning the second condition are in order:

Remark 3.33. As a projection in a Hilbert space, P_Y is directionally differentiable if U_{ad} is polyhedric, extended polyhedric, or second-order regular (see e.g. [22,34], [6, Theorem 5.5], and [7, Theorem 3.3.5, 3.3.6 and Remark 3.3.7]). As for instance proven in [48], constraint sets defined through box constraints in $L^s(\Lambda)$ and $W^{1,s}(\Lambda)$ with $s \in (1,\infty)$, i.e.,

$$\begin{aligned} U_{\mathrm{ad}}^{(0)} &:= \big\{u \in L^s(\Lambda)\colon u_{\min}(\mathrm{x}) \leq u(\mathrm{x}) \leq u_{\max}(\mathrm{x}) \text{ a.e. in } \Lambda\big\} \\ \text{and}\quad U_{\mathrm{ad}}^{(1)} &:= \big\{u \in W^{1,s}(\Lambda)\colon u_{\min}(\mathrm{x}) \leq u(\mathrm{x}) \leq u_{\max}(\mathrm{x}) \text{ a.e. in } \Lambda\big\}\end{aligned}$$

with Lebesgue-measurable functions $u_{\min}, u_{\max}\colon \Lambda \to \mathbb{R}\cup\{\pm\infty\}$ are polyhedric so that Assumption 3.32(ii) is satisfied in these prominent cases. We refer to [48] for other examples of polyhedric sets.

We now derive the regularity result. Let us first note that due to the convexity of F_λ, the solution u_λ of (P_λ) is characterized by

$$\begin{aligned}(3.29)\quad \int_J \big(\partial_t u_\lambda(t), \partial_t v(t)\big)_Y + \big\langle \varphi_{X,p}\big(u_\lambda(t) - \bar u(t)\big), v(t)\big\rangle_{X',X} \\ + \lambda^{-1}\big\langle \varphi_{X,p}\big(u_\lambda(t) - P_X(u_\lambda(t))\big), v(t)\big\rangle_{X',X}\,\mathrm{d}t \\ = \int_J \big\langle g(t), v(t)\big\rangle_{X',X}\,\mathrm{d}t \quad \text{for all } v \in \mathbb{W}^{1,2}_p(X;Y).\end{aligned}$$

It will be useful to be able to test the equation with $P_Y(u_\lambda)$ for which we need to know that this is an element of $\mathbb{W}^{1,2}_p(X;Y)$.

Lemma 3.34. *The projection P_Y maps $L^p(J;X)$ to itself. Furthermore, given $u \in W^{1,2}(J;Y)$, there holds $P_Y(u) \in W^{1,2}(J;Y)$ with*

$$(3.30)\qquad \big[\partial_t P_Y(u)\big](t) = P_Y'\big(u(t);\partial_t u(t)\big) \quad \textit{f.a.a.} t \in J.$$

Thus, P_Y maps $\mathbb{W}^{1,2}_p(X;Y)$ to $\mathbb{W}^{1,2}_p(X;Y)$.

Proof. Given a function $u \in L^p(J;X)$, the projected $P_Y(u)$ is clearly Bochner-measurable by the required continuity of P_Y. Moreover, the growth condition immediately gives $P_Y(u) \in L^p(J;X)$.

Since it is a projection in the Hilbert space Y, we know that P_Y is nonexpansive, so globally Lipschitz-continuous on Y with Lipschitz-constant 1. Now, let $u \in W^{1,2}(J;Y)$ be fixed but arbitrary. Then it holds for almost all $t \in J$ that

$$\frac{u(t+h) - u(t)}{h} \xrightarrow{h \searrow 0} \partial_t u(t) \quad \text{in } Y.$$

Together with the Lipschitz continuity and the directional differentiability of P_Y (by assumption), this yields

$$\begin{aligned}\Big\| \frac{P_Y(u(t+h)) - P_Y(u(t))}{h} - P_Y'\big(u(t);\partial_t u(t)\big)\Big\|_Y \\ \le \Big\| \frac{P_Y(u(t) + h\,\partial_t u(t)) - P_Y(u(t))}{h} - P_Y'\big(u(t);\partial_t u(t)\big)\Big\|_Y \\ + \Big\| \frac{u(t+h) - u(t) - h\,\partial_t u(t)}{h}\Big\|_Y \longrightarrow 0 \quad \text{as } h \searrow 0.\end{aligned}$$

Moreover, the Lipschitz continuity of P_Y gives for almost all $t \in J$ that

$$\begin{aligned}\big\|P_Y'\big(u(t);\partial_t u(t)\big)\big\|_Y = \lim_{h\searrow 0} \Big\| \frac{P_Y(u(t+h)) - P_Y(u(t))}{h}\Big\|_Y \\ \le \lim_{h\searrow 0} \Big\| \frac{u(t+h) - u(t)}{h}\Big\|_Y = \|\partial_t u(t)\|_Y.\end{aligned}$$

Thus, $P_Y'(u(\cdot);\partial_t u(\cdot))$ is dominated by the $L^2(J;Y)$-function $\partial_t u$. As a pointwise limit of Bochner-measurable functions, it is moreover Bochner-measurable, hence $P_Y'(u(\cdot);\partial_t u(\cdot)) \in L^2(J;Y)$. Since Y enjoys the Radon-Nikodým property—as it is a Hilbert space—, the almost everywhere existence of a "classical" derivative in $L^2(J;Y)$ implies $P_Y(u) \in W^{1,2}(J;Y)$ as claimed. ☐

Lemma 3.35. *The family $(u_\lambda)_{\lambda>0}$ is bounded in $L^p(J;X)$. Moreover, assume that $\lambda \searrow 0$. Then*

$$\begin{aligned}&u_\lambda - P_X(u_\lambda) \to 0 \quad \textit{in } L^p(J;X),\\ &u_\lambda - P_Y(u_\lambda) \to 0 \quad \textit{in } L^p(J;Y),\end{aligned} \qquad \textit{and} \qquad u_\lambda - P_Y(u_\lambda) \rightharpoonup 0 \quad \textit{in } L^p(J;X).$$

Proof. By optimality of u_λ for (P_λ) and $\bar u \in \mathcal{U}_{\mathrm{ad}}$, we have

$$\frac{1}{p}\|u_\lambda - \bar u\|^p_{L^p(J;X)} - \int_J \big\langle g(t), u_\lambda(t)\big\rangle_X \,\mathrm{d}t \le F_\lambda(u_\lambda) \\ \le F_\lambda(\bar u) = \frac{1}{2}\|\bar u\|^2_{W^{1,2}(J;Y)} - \int_J \big\langle g(t), \bar u(t)\big\rangle \,\mathrm{d}t =: C_{\bar u} < \infty.$$

Since the expression on the left hand side is a coercive function of u_λ, its boundedness implies the boundedness of $(u_\lambda)_{\lambda>0}$ in $L^p(J;X)$. Using again $F_\lambda(u_\lambda) \le F_\lambda(\bar u)$, we therefore find

$$\big\|u_\lambda - P_X(u_\lambda)\big\|^p_{L^p(J;X)} \le \frac{\lambda}{2}\Big(F_\lambda(u_\lambda) + \|g\|_{L^{p'}(J;X')}\|u_\lambda\|_{L^p(J;X)}\Big) \xrightarrow{\lambda\searrow 0} 0. \tag{3.31}$$

This immediately gives an estimate of $u_\lambda - P_Y(u_\lambda)$ in $L^p(J;Y)$:

$$\big\|u_\lambda - P_Y(u_\lambda)\big\|_{L^p(J;Y)} \le \big\|u_\lambda - P_X(u_\lambda)\big\|_{L^p(J;Y)} \le C\big\|u_\lambda - P_X(u_\lambda)\big\|_{L^p(J;X)} \to 0. \tag{3.32}$$

Due to the boundedness of $(u_\lambda)_{\lambda>0}$ in $L^p(J;X)$ and the growth condition in Assumption 3.32, the sequence $(u_\lambda - P_Y(u_\lambda))_\lambda$ is bounded in $L^p(J;X)$ and thus—possibly after passing to a subsequence—weakly converging. Thanks to (3.32), the weak limit is zero, hence unique and the whole sequence converges weakly. □

Proposition 3.36. *There holds $u_\lambda \to \bar u$ in $\mathbb{W}^{1,2}_p(X;Y)$ as $\lambda \searrow 0$.*

Proof. In (3.29), we choose $v = \bar u - u_\lambda$, and add this equality to (3.8) tested with $P_Y(u_\lambda)$, that is,

$$\int_J \big(\partial_t \bar u(t), \partial_t P_Y(u_\lambda(t)) - \partial_t \bar u(t)\big)_Y - \big\langle g(t), P_Y(u_\lambda(t)) - \bar u(t)\big\rangle_{X',X}\,\mathrm{d}t \ge 0 \quad \text{for all } v \in \mathcal{U}_{\mathrm{ad}}, \quad t \in J.$$

Note that $P_Y(u_\lambda) \in \mathbb{W}^{1,2}_p(X;Y)$ by Lemma 3.34. We obtain

$$\begin{aligned}\|\partial_t \bar u - \partial_t u_\lambda\|^2_{L^2(J;Y)} &+ \|\bar u - u_\lambda\|^p_{L^p(J;X)} \\ &\le \big\|\partial_t \bar u\big\|_{L^2(J;Y)}\big\|u_\lambda - P_Y(u_\lambda)\big\|_{L^2(J;Y)} + \int_J \big\langle g(t), u_\lambda(t) - P_Y(u_\lambda(t))\big\rangle_{X',X}\,\mathrm{d}t \\ &\quad + \lambda^{-1}\int_J \big\langle \varphi_{X,p}\big(u_\lambda(t) - P_X(u_\lambda(t))\big), \bar u(t) - P_X(u_\lambda(t))\big\rangle_{X',X}\,\mathrm{d}t \\ &\quad + \lambda^{-1}\int_J \big\langle \varphi_{X,p}\big(u_\lambda(t) - P_X(u_\lambda(t))\big), P_X(u_\lambda(t)) - u_\lambda(t)\big\rangle_{X',X}\,\mathrm{d}t.\end{aligned}$$

The two last terms on the right hand side are non-positive because of $\bar u(t) \in U_{\mathrm{ad}}$ per assumption, $\varphi_{X,p}$ being odd, and the projection variational inequality (3.17). The two other terms on the right hand side converge to zero by Lemma 3.35, which then implies $u_\lambda \to \bar u$ in $\mathbb{W}^{1,2}_p(X;Y)$ as desired. □

Lemma 3.37. *Assume that there exists a sequence $\lambda \searrow 0$ such that $\lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^{p-1}_{L^p(J;X)}$ is bounded. Then $\partial_t^2 \mathbf{u}_\lambda(\mathbf{t}) \rightharpoonup \partial_t^2 \bar{\mathbf{u}}$ in $L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ as $\lambda \searrow 0$. In particular,*

$$\partial_t^2 \bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$$

and $\partial_t \bar{\mathbf{u}}(\mathbf{T_1}) = \partial_t \bar{\mathbf{u}}(\mathbf{T_0}) = 0$.

Proof. Completely analogously to the proof of Lemma 3.20, we can apply the product rule from Theorem A.1 to (3.29) to obtain

$$\partial_t^2 \mathbf{u}_\lambda(\mathbf{t}) = -g(t) + \varphi_{X,p}\big(u_\lambda(t) - \bar u(t)\big) + \lambda^{-1}\varphi_{X,p}\big(u_\lambda(t) - P_X(u_\lambda(t))\big) \quad \text{in } X' \tag{3.33}$$

for almost all $t \in J$ along with $\partial_t^2 \mathbf{u}_\lambda \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ and $\partial_t \mathbf{u}_\lambda(\mathbf{T_0}) = \partial_t \mathbf{u}_\lambda(\mathbf{T_1}) = 0$. Since $\|\varphi_{X,p}(\cdot)\|_{X'} = \|\cdot\|^{p-1}_X$, we see that $\partial_t^2 \mathbf{u}_\lambda(\mathbf{t})$ is bounded in $L^{p'}(J;X')$ if and only if $\|u_\lambda - \bar{u}\|^{p-1}_{L^p(J;X)}$ and $\lambda^{-1}\|u_\lambda - P_X(u_\lambda)\|^{p-1}$ are bounded. Lemma 3.35 shows that the former is true, and the latter is an assumption for some sequence $\lambda \searrow 0$, so we infer that $(\partial_t^2 \mathbf{u}_\lambda(\mathbf{t}))_\lambda$ is bounded in $L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ for this sequence.

Reflexivity yields a weakly converging subsequence (which we denote by the same name) $\partial_t^2 \mathbf{u}_\lambda(\mathbf{t}) \rightharpoonup w$ in $L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$. Together with Proposition 3.36 (strong convergence $u_\lambda \to \bar{u}$ in $\mathbb{W}^{1,2}_p(X;Y)$), this implies that, for all $\phi \in C_c^\infty(J)$ and all $v \in X$,

$$\begin{aligned}\int_J \big\langle \phi'(t)v, \partial_t \bar{\mathbf{u}}(\mathbf{t})\big\rangle_{Y,Y'} \,\mathrm{d}t &= \lim_{\lambda\searrow 0} \int_J \big\langle \phi'(t)v, \partial_t \mathbf{u}_\lambda(\mathbf{t})\big\rangle_{Y,Y'} \,\mathrm{d}t \\ &= \lim_{\lambda\searrow 0} -\int_J \big\langle \phi(t)v, \partial_t^2 \mathbf{u}_\lambda(\mathbf{t})\big\rangle_{X,X'} \,\mathrm{d}t = -\int_J \big\langle \phi(t)v, w\big\rangle_{X,X'} \,\mathrm{d}t.\end{aligned}$$

Hence, $w = \partial_t^2 \bar{\mathbf{u}} \in L^{p'}(J;X') + L^{r'}(J;(Y',X')_{\eta,\infty})$ and we have $\partial_t \mathbf{u}_\lambda \rightharpoonup \partial_t \bar{\mathbf{u}}$ in $\mathbb{W}^{1,p'}_2(Y',X')$. Since the point evaluation is continuous from $\mathbb{W}^{1,p'}_2(Y',X')$ to $(Y',X')_{\frac{2}{2+p},\frac{2+p}{p}}$, this implies moreover $\partial_t \mathbf{u}_\lambda(\mathbf{T_1}) \rightharpoonup \partial_t \bar{\mathbf{u}}(\mathbf{T_1}) = 0$ and $\partial_t \mathbf{u}_\lambda(\mathbf{T_0}) \rightharpoonup \partial_t \bar{\mathbf{u}}(\mathbf{T_0}) = 0$ in the interpolation space. □

We next derive conditions which imply the validity of the assumptions that $\lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^{p-1}_{L^p(J;X)}$ be bounded in Lemma 3.37. Let us first show an auxiliary result:

Lemma 3.38. *Let $u, h \in Y$ be given. Then it holds*

$$\big(P_Y'(u;h), P_Y'(u;h) - h\big)_Y \le 0.$$

Proof. According to Assumption 3.32, P_Y is directionally differentiable in Y. By [14], the directional derivative $P_Y'(u;h) \in Y$ of P_Y at $u \in Y$ in direction $h \in Y$ is given by the unique solution $\delta \in Y$ of the following VI of the second kind:

$$\delta \in \mathcal{K}(u), \quad \big(\delta, v-\delta\big)_Y + \frac{1}{2}\, I''_{\mathrm{ad}}(u;v) - \frac{1}{2}\, I''_{\mathrm{ad}}(u;\delta) \ge \big(h, v-\delta\big)_Y \quad \forall\, v \in \mathcal{K}(u), \tag{3.34}$$

see also [7, Theorem 1.4.1] and [8]. Here, I''_{ad} denotes the weak second subderivative of the indicator functional of U_{ad} w.r.t. the topology in Y, i.e.,

$$I''_{\mathrm{ad}}(u;v) := \inf\left\{\liminf_{n\to\infty} \frac{2}{t_n}\big(P_Y(u) - u, v_n\big)_Y : t_n \searrow 0,\ P_Y(u) + t_n v_n \in U_{\mathrm{ad}},\ v_n \rightharpoonup v \text{ in } Y\right\},$$

along with the usual convention $\inf(\emptyset) = \infty$, and $\mathcal{K}(u)$ is the effective domain of $I''_{\mathrm{ad}}(u;\cdot)$, i.e., $\mathcal{K}(u) := \{v \in Y : I''_{\mathrm{ad}}(u;v) < \infty\}$. In view of the VI characterizing the projection onto U_{ad} in Y, so

$$P_Y(u) \in U_{\mathrm{ad}}, \quad \big(P_Y(u) - u, v - P_Y(u)\big)_Y \ge 0 \quad \forall\, v \in U_{\mathrm{ad}},$$

it is easily seen that $I''_{\mathrm{ad}}(u;v) \ge 0$ for all $v \in Y$. This directly implies that $I''_{\mathrm{ad}}(u;0) = 0$ (so that $0 \in \mathcal{K}(u)$). Therefore, by testing (3.34) with $v = 0$, we obtain

$$(\delta - h, -\delta)_Y \ge \frac{1}{2}\, I''_{\mathrm{ad}}(u;\delta) \ge 0.$$

Thanks to $\delta = P_Y'(u;h)$, this is the assertion. □

We frequently use that $\varphi_{X,p}$ is an odd function in the following. From Lemma 3.34 we know that $P_Y(u_\lambda) \in \mathbb{W}^{1,2}_p(X;Y)$ so that we are allowed to insert $u_\lambda - P_Y(u_\lambda)$ as the test function in (3.29) to obtain together with (3.30) and Lemma 3.38 (with $u = u_\lambda$ and $h = \partial_t u_\lambda$) that

$$\begin{aligned}&\big\|\partial_t u - \partial_t(P_Y(u))\big\|^2_{L^2(J;Y)} \\ &\qquad + \lambda^{-1} \int_J \big\langle \varphi_{X,p}\big(u_\lambda(t) - P_X(u_\lambda(t))\big), u_\lambda(t) - P_Y(u_\lambda(t))\big\rangle_{X',X} \,\mathrm{d}t \\ &\quad \le \int_J \big\langle g(t), u_\lambda(t) - P_Y(u_\lambda(t))\big\rangle_{X',X} + \big\langle \varphi_{X,p}\big(u_\lambda(t) - \bar{u}(t)\big), P_Y(u_\lambda(t)) - u_\lambda(t)\big\rangle_{X',X} \,\mathrm{d}t.\end{aligned} \tag{3.35}$$

The definition of P_X as a projection operator implies (cf. (3.17)) that

$$\big\langle \varphi_{X,p}\big(P_X(w)-u\big), v-P_X(w)\big\rangle_{X',X} \geq 0 \quad \text{for all } v\in U_{\mathrm{ad}}, \quad w\in X. \tag{3.36}$$

Since $P_Y(w)\in X$ by assumption, we may insert $v=P_Y(w)$ and rearrange to obtain

$$\big\langle \varphi_{X,p}\big(P_X(w)-w\big), P_Y(w)-w\big\rangle_{X',X} \geq \big\langle \varphi_{X,p}\big(P_X(w)-w\big), P_X(w)-w\big\rangle_{X',X} = \|P_X(w)-w\|_X^p$$

for all $w\in X$. Using this for the second addend on the left hand side of (3.35) and neglecting the first one, we find

$$\begin{aligned}\lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^p_{L^p(J;X)}\\ \leq \int_J \big\langle g(t), u_\lambda(t)-P_Y(u_\lambda(t))\big\rangle_{X',X} + \big\langle \varphi_{X,p}\big(u_\lambda(t)-\bar u(t)\big), P_Y(u_\lambda(t))-u_\lambda(t)\big\rangle_{X',X}\,\mathrm{d}t.\end{aligned} \tag{3.37}$$

Using Lemma 3.35, we easily see that the right hand side is bounded. Unfortunately, the power p on the left hand side is wrong: Lemma 3.37 needs that $\lambda^{-1}\|u_\lambda - P_X(u_\lambda)\|^{p-1}_{L^p(J;X)}$ is bounded, and due to $u_\lambda - P_X(u_\lambda)\to 0$ in $L^p(J;X)$ as $\lambda\searrow 0$, we have

$$\lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^p_{L^p(J;X)} \leq \lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^{p-1}_{L^p(J;X)} \quad \text{as } \lambda\searrow 0.$$

Hence, an estimate of the right hand side with a factor $\|u_\lambda - P_X(u_\lambda)\|_{L^p(J;X)}$ seems indispensable. This we achieve as follows:

Lemma 3.39. *Assume that the projection P_Y is Lipschitz continuous as a mapping on X. Then*

$$\limsup_{\lambda\searrow 0} \lambda^{-1}\big\|u_\lambda - P_X(u_\lambda)\big\|^{p-1}_{L^p(J;X)} < \infty.$$

Proof. We estimate the right hand side in (3.37) by

$$(3.37) \quad \leq \big(\|g\|_{L^{p'}(J;X')} + \|u_\lambda - \bar u\|^{p-1}_{L^p(J;X)}\big)\big\|u_\lambda - P_Y(u_\lambda)\big\|_{L^p(J;X)}.$$

Inserting $P_X(u)$ and using the assumed Lipschitz continuity of P_Y in X and that $P_X(u_\lambda(t))\in U_{\mathrm{ad}}$, we obtain

$$\begin{aligned}\big\|u_\lambda - P_Y(u_\lambda)\big\|_{L^p(J;X)} &\leq \big\|u_\lambda - P_X(u_\lambda)\big\|_{L^p(J;X)} + \big\|P_X(u_\lambda)-P_Y(u_\lambda)\big\|_{L^p(J;X)}\\ &= \big\|u_\lambda - P_X(u_\lambda)\big\|_{L^p(J;X)} + \big\|P_Y\big(P_X(u_\lambda)\big)-P_Y(u_\lambda)\big\|_{L^p(J;X)}\\ &\leq (1+L)\big\|u_\lambda - P_X(u_\lambda)\big\|_{L^p(J;X)},\end{aligned}$$

where $L>0$ is the Lipschitz constant of P_Y as a mapping on X. This implies the claim by re-insertion into (3.37) and Lemma 3.35. ☐

Remark 3.40. We emphasize that the Lipschitz assumption on P_Y in Lemma 3.39 is strictly stronger than Assumption 3.32(i) and *very* restrictive. For example, it is in general *not* satisfied for $Y=L^2(\Lambda)$, $X=H_0^1(\Lambda)$ and $U_{\mathrm{ad}}=\{u\in L^2(\Lambda)\colon u\geq 0 \text{ a.e. in } \Lambda\}$. See Lemma B.1 for an explicit counterexample for $\Lambda=(0,1)$. Nevertheless, there are of course examples where such a Lipschitz continuity holds, most prominently $Y=L^2(\Lambda)$, $X=L^s(\Lambda)$ for some $s>2$, and box constraints U_{ad}, since the projections there agree as we have seen in Remark 3.26. (In this case, we could also just replace $P_Y(u_\lambda(t))$ by $P_X(u_\lambda(t))$ in (3.35) and be nearly done.)

Summing up, we have just proven the following:

Proposition 3.41. *In the setting of Section 3.3.2, suppose in addition that Assumption 3.32(ii) is satisfied and that P_Y is Lipschitz continuous on X. Then Assumption 3.27(i) is satisfied.*

Example (continued)**.** We verify the sufficient conditions for Assumption 3.27(i) to hold for the running example. It was $Y = L^2(\partial\Omega)$ and $X = L^p(\partial\Omega)$ with $p \geq 2$ and U_{ad} given by box constraints with limit functions $u_{\min}, u_{\max} \in L^p(\partial\Omega)$. As noted in Remark 3.33, Assumption 3.32(ii) is thus satisfied. Regarding the Lipschitz continuity, as seen in Remark 3.26 the metric projections onto U_{ad} in $X = L^p(\partial\Omega)$ and $Y = L^2(\Omega)$ coincide and are given by the pointwise projections in $\mathbb{R}$ onto $[u_{\min}(\mathrm{x}), u_{\max}(\mathrm{x})]$ for almost every $x \in \partial\Omega$. This makes this projection in particular Lipschitz continuous, also in $L^p(\partial\Omega)$. Thus for (Ex-OCP) we can refer to Proposition 3.41 to find Assumption 3.27(i) to be satisfied.

4. Examples

We close this work by giving further examples for constellations of spaces $\mathcal{U}_w$ and X, Y such that Theorem 2.9 is applicable, and how to apply the results from §2 in these cases in accordance with the assumptions in §3.

4.1. **Preliminaries.** The examples displayed in the following affect evolution equations which act on a *bounded* domain $\Omega \subset \mathbb{R}^d$. We suppose $d \geq 2$ to avoid particularities in the Sobolev embeddings and related topics below. In the sequel, we demand for Ω the *Lipschitz* property in the spirit of [19, Def. 1.2.1.2], cf. also [31, Ch. 1.1.9, Def. 3]. Note that this class of domains properly includes *strong* Lipschitz domains, cf. [31, Ch. 1.1.9, Def. 1] or [19, Def. 1.2.1.2]. Lipschitz domains have the advantage that they are, on the one hand, fairly general, such that they cover almost everything what appears in real world applications. On the other hand, function spaces on Lipschitz domains enjoy many of the properties which are well-known in case of smooth domains. For the convenience of the reader we recall some basic facts on bounded Lipschitz domains and on first order Sobolev spaces on these domains which are needed later.

(i) There is a continuous extension operator $\mathfrak{E} : L^1(\Omega) \to L^1(\mathbb{R}^d)$ which also maps $L^s(\Omega)$ continuously into $L^s(\mathbb{R}^d)$ if $s \in (1, \infty]$. Even more, the restriction of $\mathfrak{E}$ to $W^{1,q}(\Omega)$ provides a continuous mapping into $W^{1,q}(\mathbb{R}^d)$ where $q \in [1, \infty)$, cf. [17, p. 165].

(ii) From the first point, one may deduce all the usual Sobolev embeddings for spaces which include first order derivatives, including compactness properties.

(iii) The set of restrictions of $C_c^\infty(\mathbb{R}^d)$-functions to Ω is dense in any space $W^{1,q}(\Omega)$ as long as $q \in [1, \infty)$.

(iv) Using the bi-Lipschitzian boundary charts implied by the definition of Lipschitz domains, it is not hard to see that the boundary measure σ on $\partial\Omega$ satisfies the following property: For any point $\mathrm{x} \in \partial\Omega$ and any ball $B(x, \rho)$ around x with radius $\rho \in (0, 1)$, one obtains the estimate

$$c_0 \rho^{d-1} \leq \sigma\big(\partial\Omega \cap B(\mathrm{x}, \rho)\big) \leq c_1 \rho^{d-1} \tag{4.1}$$

for two positive constants c_0, c_1, independent of x and ρ, compare [16, Ch. 3.3.4C] and [21, Ch. 3.1].

We proceed to give examples for control spaces $\mathcal{U}_w$ and suitable spaces X with integrability indices p such that the assumptions of Theorem 2.9 and the ones in §3 are satisfied. We concentrate on the space $Y = L^2(\Lambda)$ for a suitable set Λ in these cases, since, as already sketched in the introduction and in Section 3, it is a central point to take Y as a Hilbert space. In the examples, we follow the following rough roadmap, given the spatial "target space" U:

(i) identify smooth reflexive spaces X which embed densely into Y and such that there exists a compact linear operator $E \in \mathcal{K}(X; U)$,

(ii) determine $\eta \in (0, 1)$ such that $E \in \mathcal{L}((X, Y)_{\eta,1}; U)$

(iii) give conditions on $p \in (1, \infty)$ such that Theorem 2.9 is applicable.

The examples which we consider for U are $C(\overline{\Omega})$ in §4.2, $L^s(\Omega)$ in §4.3, and $W_0^{-1,q}(\Omega)$ for both distributed and boundary control in §4.4. Thereby, examples 1 and 2 in sections 4.2 and 4.3 fit in the classical Aubin-Lions setting with the form $X \hookrightarrow U \hookrightarrow Y$, whereas example 3 in §4.4 does not.

4.2. **Example 1:** We consider the case $U = C(\overline{\Omega})$, so e.g. $\mathcal{U}_w = C(J; C(\overline{\Omega}))$ or $L^\infty(J; L^\infty(\Omega))$. Let us give a brief motivation:

Uniformly continuous functions for $\mathcal{U}_w$ could for instance occur in a setting of *optimal control of coefficients* or parameter identification (inverse problems) for a PDE of the form

$$\partial_t w(t) - \nabla \cdot u(t) \nabla w(t) = f(t), \quad w(T_0) = w_0$$

with $f(t) \in W_0^{-1,q}(\Omega)$ for $q > d$. (The latter could correspond e.g. to inhomogeneous Neumann boundary data.) The goal is to identify or optimize the coefficient function u; a lower bound in the form of a control constraint $u(t, \mathrm{x}) \geq u_0 > 0$ is also natural. Since the equation is nonautonomous, it will be extremely useful to have constant domains for the differential operator $-\nabla \cdot u(t)\nabla$ for every t and every feasible choice $u \in \mathcal{U}_w$ with a continuous dependence on t, cf. [39]. From [13, Thm. 6.3] it is known that this is the case for $\mathcal{U}_w = C(J; C(\overline{\Omega}))$ whenever $-\Delta$ is an isomorphism between $W^{1,q}(\Omega)$ and $W_0^{-1,q}(\Omega)$.

See moreover [36, 37, 40] for examples where the $L^\infty(J; L^\infty(\Omega))$ regularity was needed for continuity of the control-to-state operator, see also [18] for an example with $\mathcal{U} = L^\infty(J; L^\infty(\partial\Omega))$.

We set up Theorem 2.9 for $U = C(\overline{\Omega})$ and $\varrho = 0$. The obvious smooth and reflexive spaces X and mappings $E \in \mathcal{K}(X; U)$ are $X = W^{1,q}(\Omega)$ and the associated embeddings $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$, so we have

$$U = C(\overline{\Omega}), \quad X = W^{1,q}(\Omega) \text{ with } q > d, \quad Y = L^2(\Omega) \quad \text{and} \quad E = \mathrm{id}.$$

Of course, $X \hookrightarrow Y$ densely. Before we give the result, we need an auxiliary statement which is a classical one from interpolation theory:

Lemma 4.1 ([46, Lem. 1.10.1])**.** *Let X, Y, Z be Banach spaces with $X \hookrightarrow Z \hookrightarrow Y$, let $\theta \in (0, 1)$, and let for every $f \in X$ the following inequality be true:*

$$\|f\|_Z \leq C \|f\|_X^{1-\theta} \|f\|_Y^\theta.$$

Then one has the continuous embedding $(X, Y)_{\theta,1} \hookrightarrow Z$.

Here is the result for this example:

Lemma 4.2. *Let $q > d$ and $r \in [1, \infty)$ and set*

$$\bar{\eta} = \frac{1 - \frac{d}{q}}{1 - \frac{d}{q} + \frac{d}{2}}.$$

(i) *Let $p > \frac{2(1-\bar{\eta})r}{2+\bar{\eta}r}$. Then*

$$\mathbb{W}_p^{1,2}\big(W^{1,q}(\Omega), L^2(\Omega)\big) \hookrightarrow\hookrightarrow L^r\big(J; C(\overline{\Omega})\big).$$

(ii) *Let $p > \frac{2(1-\bar{\eta})}{\bar{\eta}} = \frac{qd}{q-d}$. Then*

$$\mathbb{W}_p^{1,2}\big(W^{1,q}(\Omega), L^2(\Omega)\big) \hookrightarrow\hookrightarrow C\big(J; C(\overline{\Omega})\big).$$

Proof. Let $\eta \in (0, 1)$. We have $(W^{1,q}, L^2(\Omega))_{\eta,1} \hookrightarrow (W^{1,q}(\Omega), L^2(\Omega))_{\eta,\tau}$ for all $\tau \in [1, \infty]$, see [46, Ch. 1.3.3]. Moreover, by [46, Ch. 2.4.2, (9)], we know that

$$\big(W^{1,q}(\mathbb{R}^d), L^2(\mathbb{R}^d)\big)_{\eta,\tau} = B^{\varsigma}_{\tau,\tau}(\mathbb{R}^d), \quad \text{where} \quad \varsigma = 1 - \eta \quad \text{and} \quad \frac{1}{\tau} = \frac{1-\eta}{q} + \frac{\eta}{2}$$

and $B^{\varsigma}_{\tau,\tau}(\mathbb{R}^d)$ denotes the usual Besov space. Moreover, $B^{\varsigma}_{\tau,\tau}(\mathbb{R}^d) \hookrightarrow C^\alpha(\mathbb{R}^d)$ if $\alpha := \varsigma - \frac{d}{\tau} > 0$, cf. [46, Ch. 2.8.1]. Together, we obtain $(W^{1,q}(\mathbb{R}^d), L^2(\mathbb{R}^d))_{\eta,\tau} \hookrightarrow C^\alpha(\mathbb{R}^d)$ if

$$\alpha = 1 - \eta - d\left(\frac{1-\eta}{q} + \frac{\eta}{2}\right) > 0 \quad\iff\quad \eta < \bar\eta = \frac{1-\frac{d}{q}}{1-\frac{d}{q}+\frac{d}{2}}. \tag{4.2}$$

We employ the extension operator $\mathfrak{E}$ to transfer the function space relations on $\mathbb{R}^d$ to Ω as follows. Let η satisfy (4.2) and set $\tau = (\frac{1-\eta}{q} + \frac{\eta}{2})^{-1}$. Consider $u \in (W^{1,q}(\Omega), L^2(\Omega))_{\eta,\tau}$. With $s = 1-\eta$ and $\alpha = s - \frac{d}{\tau}$, we have

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\|\mathfrak{E}u\|_{C^\alpha(\mathbb{R}^d)} \le C\|\mathfrak{E}u\|^{1-\eta}_{W^{1,q}(\mathbb{R}^d)}\|\mathfrak{E}u\|^{\eta}_{L^2(\mathbb{R}^d)} \le C\|u\|^{1-\eta}_{W^{1,q}(\Omega)}\|u\|^{\eta}_{L^2(\Omega)},$$

where the embedding in the middle follows from $(W^{1,q}(\mathbb{R}^d), L^2(\mathbb{R}^d))_{\eta,\tau} \hookrightarrow C^\alpha(\mathbb{R}^d)$ due to the choice of η. From the preceding inequality it follows $(W^{1,q}(\Omega), L^2(\Omega))_{\eta,1} \hookrightarrow C^\alpha(\overline{\Omega})$, cf. Lemma 4.1, and this clearly implies the desired (compact) embedding $(W^{1,q}(\Omega), L^2(\Omega))_{\eta,1} \hookrightarrow C(\overline{\Omega})$.

It remains to identify p in dependence of r so that η can be chosen such that Theorem 2.9 can be used. For $r \in [1,\infty)$, this is $\frac{1}{r} > \frac{1-\bar\eta}{p} - \frac{\bar\eta}{2}$. Rearranging yields

$$p > \frac{2(1-\bar\eta)r}{2+\bar\eta r}.$$

For the second case, corresponding to $r = \infty$, we need to determine p such that $\eta \in (\frac{2}{2+p}, 1)$ is feasible. From (4.2), this is the case if and only if

$$\frac{2}{2+p} < \frac{1-\frac{d}{q}}{1-\frac{d}{q}+\frac{d}{2}} \quad\iff\quad p > \frac{qd}{q-d}.$$

The claim follows then by Theorem 2.9. □

Remark 4.3. One actually obtains even an embedding into the space of Hölder continuous functions in the second case in setting of Lemma 4.2, as one sees from the proof and Theorem 2.9. We have left out the details for the sake of simplicity at this point.

4.3. **Example 2:** Let us next consider the most classical case $U = L^s(\Omega)$, so $\mathcal{U}_w = L^r(J; L^s(\Omega))$ or $C(J; L^s(\Omega))$, with $r \in [1,\infty]$ and $s \in (2,\infty)$. The case $s \in (1,2]$ follows of course via embedding. This is one of the standard situations in optimal control of PDEs.

In this example, we choose a space of type $W^{1,q}(\Omega)$ for X as in Example 1, but require only $1 - \frac{d}{q} + \frac{d}{s} > 0$, because by the Rellich-Kondrachov theorem this is enough to ensure that the embedding E satisfies $X = W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^s(\Omega) = U$. From this it of course also follows that $1 - \frac{d}{q} + \frac{d}{2} > 0$ with the analogous meaning $X = W^{1,q}(\Omega) \hookrightarrow L^2(\Omega) = Y$. So, here we have

$$U = L^s(\Omega), \quad X = W^{1,q}(\Omega) \text{ with } q \le d, \quad Y = L^2(\Omega) \quad\text{and}\quad E = \mathrm{id}.$$

Note that we assume $q \le d$ in this example, since the reasoning for $q > d$ is already laid out in Example 1, §4.2, where we had $1 - \frac{d}{q} > 0$ as the supposition on q.

Lemma 4.4. *Let $q \le d$ and $r \in [1,\infty)$ and $s > 2$, and let $1 - \frac{d}{q} + \frac{d}{s} > 0$. Moreover, set*

$$\bar\eta = \frac{1-\frac{d}{q}+\frac{d}{s}}{1-\frac{d}{q}+\frac{d}{2}}.$$

Then we have the following embeddings:

(i) *If $p > \frac{2(1-\bar\eta)r}{2+\bar\eta r}$, then*

$$\mathbb{W}^{1,2}_p\big(W^{1,q}(\Omega), L^2(\Omega)\big) \hookrightarrow\hookrightarrow L^r\big(J; L^s(\Omega)\big).$$

(ii) *If* $p > \frac{2(1-\bar\eta)}{\bar\eta}$, *then*

$$\mathbb{W}_p^{1,2}\big(W^{1,q}(\Omega), L^2(\Omega)\big) \hookrightarrow\hookrightarrow C\big(J; L^s(\Omega)\big).$$

Proof. We use the *Gagliardo-Nirenberg inequality* [38, Lect. II, Thm.] to obtain the embeddings on $\mathbb{R}^n$. Observe that

$$\bar\eta = \bar\eta(s,q) = \frac{1 - \frac{d}{q} + \frac{d}{s}}{1 - \frac{d}{q} + \frac{d}{2}} \qquad \iff \qquad \frac{d}{s} = (1-\bar\eta)\left(\frac{d}{q} - 1\right) + \bar\eta \cdot \frac{d}{2},$$

hence by the Gagliardo-Nirenberg inequality,

$$\|w\|_{L^s(\mathbb{R}^d)} \le C\|w\|^{1-\bar\eta}_{W^{1,q}(\mathbb{R}^d)}\|w\|^{\bar\eta}_{L^2(\mathbb{R}^d)} \quad \text{for all } w \in W^{1,q}(\mathbb{R}^d).$$

Employing again the extension operator $\mathfrak{E}$, the preceding inequality extends to

$$\|u\|_{L^s(\Omega)} \le \|\mathfrak{E}u\|_{L^s(\mathbb{R}^d)} \le C\|\mathfrak{E}u\|^{1-\bar\eta}_{W^{1,q}(\mathbb{R}^d)}\|\mathfrak{E}u\|^{\bar\eta}_{L^2(\Omega)} \le C\|u\|^{1-\bar\eta}_{W^{1,q}(\Omega)}\|u\|^{\bar\eta}_{L^2(\Omega)}$$

for all $u \in W^{1,q}(\Omega)$, or equivalently, per Lemma 4.1, $(W^{1,q}(\Omega), L^2(\Omega))_{\bar\eta,1} \hookrightarrow L^s(\Omega)$.

As in the proof of Lemma 4.2, we next determine the necessary magnitude of p in order to have $\frac{1}{r} > \frac{1-\bar\eta}{p} - \frac{\bar\eta}{2}$ as required in Theorem 2.9 for the embedding into $L^r(J; L^s(\Omega))$. Rearranging yields again

$$p > \frac{2(1-\bar\eta)r}{2+\bar\eta r}.$$

For the embedding into $C(J; L^s(\Omega))$, in order to obtain $\bar\eta > \frac{2}{2+p}$, we need

$$p > 2 \cdot \frac{\frac{d}{2} - \frac{d}{s}}{1 - \frac{d}{q} + \frac{d}{s}} = \frac{2(1-\bar\eta)}{\bar\eta}$$

which is as expected exactly the limit of the $L^r(J; L^s(\Omega))$ case as $r \to \infty$. With the assumptions on p, s and q, Theorem 2.9 then gives the assertion. ☐

Remark 4.5. As in Remark 4.3, the foregoing lemma in fact yields an embedding into the Hölder space $C^\varrho(J; L^s(\Omega))$ in the second case. Here one calculates directly that the Hölder order ϱ is restricted by $\varrho < \frac{\bar\eta}{2} - \frac{1-\bar\eta}{p}$ for $p > \frac{2(1-\bar\eta)}{\bar\eta}$.

4.4. **Example 3.** Let us finally consider negative Sobolev spaces $U = W_0^{-1,q}(\Omega)$, the dual space of $W^{1,q'}(\Omega)$. These can occur for both distributed controls, so functions actually acting on (parts of) Ω, and boundary control, so controls which act only on (parts of) the boundary $\partial\Omega$. Both settings, in particular the latter, are somewhat different in nature than the preceding ones since they take full advantage of the admissible setting in Theorem 2.9: The spaces $W_0^{-1,q}(\Omega)$ do not embed into the Hilbert pivot spaces $L^2(\Omega)$ or $L^2(\partial\Omega)$ at all, so we do not have a classical $X \hookrightarrow Z \hookrightarrow Y$ type structure. Also, for the boundary functions, the mapping E is not derived from an embedding, but from the (non-injective!) trace operator. Generally, the negative Sobolev setting is the other standard setup for optimal control of PDEs and found in countless contributions.

4.4.1. *Distributed control.* We begin with the "distributed control" setting, so $X = L^s(\Omega)$ and $Y = L^2(\Omega)$. Of course, we suppose $s > 2$. We further consider $q > \frac{2d}{d-2}$ since otherwise $L^2(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, but exclude $q = \infty$. Note that this implicitly means $d \ge 3$. For $s > \frac{dq}{q+d}$, the Rellich-Kondrachov theorem yields $L^s(\Omega) \hookrightarrow\hookrightarrow W_0^{-1,q}(\Omega)$. So, we collect

$$U = W_0^{-1,q}(\Omega) \text{ with } q > \frac{2d}{d-2}, \quad X = L^s(\Omega) \text{ with } s > \frac{dq}{q+d}, \quad Y = L^2(\Omega) \quad \text{and} \quad E = \mathrm{id}.$$

We have the following result:

Lemma 4.6. *Let $q > \frac{2d}{d-2}$ and $r \in [1,\infty)$ and $s > \frac{dq}{q+d}$. Set*

$$\bar\eta = \frac{\frac{1}{q} + \frac{1}{d} - \frac{1}{s}}{\frac{1}{2} - \frac{1}{s}}.$$

Then we have the following embeddings:

(i) *If $p > \frac{2(1-\bar\eta)r}{2+\bar\eta r}$, then*

$$\mathbb{W}^{1,2}_p\big(L^s(\Omega), L^2(\Omega)\big) \hookrightarrow L^r\big(J; W_0^{-1,q}(\Omega)\big).$$

(ii) *If $p > \frac{2(1-\bar\eta)}{\bar\eta}$, then*

$$\mathbb{W}^{1,2}_p\big(L^s(\Omega), L^2(\Omega)\big) \hookrightarrow C\big(J; W_0^{-1,q}(\Omega)\big).$$

Proof. Let $\eta \in (0,1)$. From [46, Ch. 1.10.1/3 and Ch. 1.18.4], we obtain

$$\big(L^s(\Omega), L^2(\Omega)\big)_{\eta,1} \hookrightarrow \big[L^s(\Omega), L^2(\Omega)\big]_\eta = L^\varsigma(\Omega)$$

with $\frac{1}{\varsigma} = \frac{1-\eta}{s} + \frac{\eta}{2}$. Hence, $(L^s(\Omega), L^2(\Omega))_{\eta,1} \hookrightarrow W_0^{-1,q}(\Omega)$ via $L^\varsigma(\Omega)$ if $\varsigma \geq \frac{dq}{q+d}$, or equivalently,

$$\eta \leq \bar\eta := \frac{\frac{1}{q} + \frac{1}{d} - \frac{1}{s}}{\frac{1}{2} - \frac{1}{s}}.$$

From here, we proceed as in the proof of Lemma 4.4. □

4.4.2. *Boundary control.* This particular setting is taken from the optimal control of the thermistor problem, cf. [25, 32, 33] and the running example: The control is acting on (a subset of) the boundary and thus induces a bounded linear functional on spaces of type $W^{1,q'}(\Omega)$. Moreover, the general setting also includes mixed boundary conditions, which we incorporate for the sake of generality. Therefore, we introduce the following spaces:

Definition 4.7. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let D be a closed subset of $\partial\Omega$. For $q \in [1,\infty)$, we define $W^{1,q}_D(\Omega)$ as the completion of $C^\infty_D(\Omega) := \{\psi_{\restriction\Omega} \colon \psi \in C_0^\infty(\mathbb{R}^d),\ \mathrm{supp}(\psi) \cap D = \emptyset\}$ with respect to the norm $\psi \mapsto \big(\int_\Omega |\psi|^q + |\nabla\psi|^q \,\mathrm{dx}\big)^{\frac{1}{q}}$. The dual space of $W^{1,q'}_D(\Omega)$ is denoted by $W^{-1,q}_D(\Omega)$.

Now let us assume that $\mathcal{U} = L^\infty(J; W^{-1,q}_D(\Omega))$ or even $\mathcal{U} = C(J; W^{-1,q}_D(\Omega))$. Again, in view of Assumption 3.15, $\mathcal{U} = L^r(J; W^{-1,q}_D(\Omega))$ for r arbitrarily large is also particularly interesting. In any case, we choose $U = W^{-1,q}_D(\Omega)$ and plan to use $X = L^s(\Gamma)$ for suitable s to be specified below. We give two auxiliary results for the (adjoint) trace operator, cf. also the running example. The first one is the main result in [5]; its assumptions are satisfied for our Lipschitz domain setting as explained in §4.1, in particular (4.1) is important.

Lemma 4.8 ([5])**.** *Let $q' \in (1,d)$ and $s' \in [1, q'\frac{d-1}{d-q'}]$. Then the trace operator* tr *maps $W^{1,q'}(\Omega)$ continuously to $L^{s'}(\partial\Omega)$. If $s' \in [1, q'\frac{d-1}{d-q'})$, then we even have* $\mathrm{tr} \in \mathcal{K}(W^{1,q'}(\Omega); L^{s'}(\partial\Omega))$.

We set $\Gamma := \partial\Omega \setminus D$.

Corollary 4.9. *Let $q > \frac{d}{d-1}$ and $s \in [q\frac{d-1}{d}, \infty]$. Then $E = \mathrm{tr}^*$ maps $L^s(\Gamma)$ continuously to $W^{-1,q}_D(\Omega)$ and even compactly if $s > q\frac{d-1}{d}$.*

Proof. We take the adjoint operator in Lemma 4.8 which gives $\mathrm{tr}^* \colon L^s(\partial\Omega) \to W^{-1,q}(\Omega)$ for the stated ranges of q and s. The claim then follows from $L^s(\Gamma) \hookrightarrow L^s(\partial\Omega)$ by extension by zero and $W^{-1,q}(\Omega) \hookrightarrow W^{-1,q}_D(\Omega)$ by restriction to the subspace $W^{1,q'}_D(\Omega)$. □

Now, let us assume that $q \geq \frac{2d}{d-1}$, i.e., tr^* is *not* a compact linear operator from $L^2(\Gamma)$ to $W^{-1,q}_D(\Omega)$. Otherwise, the time-extension $\mathcal{E}$ of tr^* would already map $W^{1,2}(J;L^2(\Gamma))$ compactly to $C(J;W^{-1,q}_D(\Omega))$. An important case for this setting is $q > d = 3$, see the running example throughout the work.

So, we choose

$$U = W^{-1,q}_D(\Omega) \text{ with } q \geq \frac{2d}{d-1}, \quad X = L^s(\Gamma) \text{ with } s > q\,\frac{d-1}{d}, \quad Y = L^2(\Gamma) \quad \text{and} \quad E = \mathrm{tr}^*.$$

By Corollary 4.9 we have $E \in \mathcal{K}(X;U)$, and the assumptions on s and q imply $s > 2$ such that $X \hookrightarrow Y$ densely.

Lemma 4.10. *Let $q \geq \frac{2d}{d-1}$ and $r \in [1,\infty)$ and $s > q\,\frac{d-1}{d}$. Set*

$$\bar\eta := \frac{\frac{d}{q(d-1)} - \frac{1}{s}}{\frac{1}{2} - \frac{1}{s}}$$

Then the time extension $\mathcal{E}$ of the adjoint trace operator $E = \mathrm{tr}^*$ has the following properties:

(i) *If $p > \frac{2(1-\bar\eta)r}{2+\bar\eta r}$, then*

$$\mathcal{E} \in \mathcal{K}\big(\mathbb{W}^{1,2}_p\big(L^s(\Gamma), L^2(\Gamma)\big); L^r\big(J; W^{-1,q}_D(\Omega)\big)\big).$$

(ii) *If $p > \frac{2(1-\bar\eta)}{\bar\eta}$, then*

$$\mathcal{E} \in \mathcal{K}\big(\mathbb{W}^{1,2}_p\big(L^s(\Gamma), L^2(\Gamma)\big); C\big(J; W^{-1,q}_D(\Omega)\big)\big).$$

Proof. Let $\eta \in (0,1)$. As in the proof of Lemma 4.6, we obtain

$$\big(L^s(\Gamma), L^2(\Gamma)\big)_{\eta,1} \hookrightarrow \big[L^s(\Gamma), L^2(\Gamma)\big]_\eta = L^\varsigma(\Omega)$$

with $\frac{1}{\varsigma} = \frac{1-\eta}{s} + \frac{\eta}{2}$. Hence, tr^* maps $(L^s(\Gamma), L^2(\Gamma))_{\eta,1}$ continuously into $W^{-1,q}_D(\Omega)$ via $L^\varsigma(\Gamma)$ if $\varsigma \geq q\frac{d-1}{d}$, or equivalently,

$$\eta \leq \bar\eta := \frac{\frac{d}{q(d-1)} - \frac{1}{s}}{\frac{1}{2} - \frac{1}{s}}.$$

From here, we again proceed as in the proof of Lemma 4.4. □

Remark 4.11. As it occurred in the running example, let us point out that at least for the second case, so formally $r = \infty$, the choice $s = p$ in the foregoing result is valid and yields the—slightly more reasonable—condition $p > \frac{2q(d-1)}{d} - 2$. Of course, in the second case, we also have again an embedding into a Hölder space, cf. Remark 4.3, and this also holds for Lemma 4.6.

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Appendix A. Integration by parts

As a foregoing remark, let us note that for $\tau, \varsigma \in (1,\infty)$, we have

$$W^{1,\tau}(J;F) \cap L^\varsigma(J;E) = \mathbb{W}^{1,\tau}_\varsigma(E,F) \hookrightarrow C(J;(E,F)_{\xi,\frac{1}{\xi}}) \tag{A.1}$$

where $\xi = \frac{1}{\tau}(1+\frac{1}{\tau}-\frac{1}{\varsigma})^{-1}$ and E and F are Banach spaces. This follows from [46, Ch. 1.8.3/1.11.2], see also Lemma 2.7.

Theorem A.1. *Let E_i, F_i be reflexive Banach spaces with $E_i \hookrightarrow F_i$ densely such that $E_1, \ldots, E_m$ and $F_1, \ldots, F_m$ are compatible, and let $\tau_i, \varsigma_i \in (1, \infty)$, all for $i = 1, \ldots, m$. Set $E = \cap_{i=1}^m E_i$ and $F = \sum_{i=1}^m F_i$ as well as $\mathbb{W}^{1,\tau}_\varsigma(E,F) = \cap_{i=1}^m \mathbb{W}^{1,\tau_i}_{\varsigma_i}(E_i,F_i)$ and $\mathbb{W}^{1,\varsigma'}_{\tau'}(F',E') = \sum_{i=1}^m \mathbb{W}^{1,\varsigma_i'}_{\tau_i'}(F_i',E_i')$. Suppose that $u \in \mathbb{W}^{1,\tau}_\varsigma(E,F)$ and $v \in \mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$. Then for every $t \in J$ we have*

$$\int_{T_0}^t \big\langle \partial_s u(s), v(s) \big\rangle_{F,F'} + \big\langle u(s), \partial_s v(s) \big\rangle_{E,E'} \, \mathrm{d}s = \big\langle u(t), v(t) \big\rangle_\xi - \big\langle u(T_0), v(T_0) \big\rangle_\xi, \tag{A.2}$$

where $\langle \cdot, \cdot \rangle_\xi$ denotes the duality pairing between the spaces $\cap_{i=1}^m (E_i, F_i)_{\xi_i, \frac{1}{\xi_i}}$ and $\sum_{i=1}^m (E_i,F_i)'_{\xi_i,\frac{1}{\xi_i}}$ with $\xi_i = \frac{1}{\tau_i}(1 + \frac{1}{\tau_i} - \frac{1}{\varsigma_i})^{-1}$.

Proof. We fall back to the fundamental theorem of calculus. Let us recall from (A.1) that

$$\mathbb{W}^{1,\tau_i}_{\varsigma_i}(E_i,F_i) \hookrightarrow C(J; (E_i,F_i)_{\xi_i,\frac{1}{\xi_i}}) \quad \text{and} \quad \mathbb{W}^{1,\varsigma_i'}_{\tau_i'}(F_i',E_i') \hookrightarrow C(J; (F_i',E_i')_{1-\xi_i,\frac{1}{1-\xi_i}}),$$

and $(F_i',E_i')_{1-\xi_i,\frac{1}{1-\xi_i}} = (E_i',F_i')_{\xi_i,\frac{1}{1-\xi_i}} = (E_i,F_i)'_{\xi_i,\frac{1}{\xi_i}}$. Thus,

$$\mathbb{W}^{1,\tau}_\varsigma(E,F) \hookrightarrow C\Big(J; \bigcup_{i=1}^m (E_i,F_i)_{\xi_i,\frac{1}{\xi_i}}\Big)$$

and

$$\mathbb{W}^{1,\varsigma_i'}_{\tau_i'}(F_i',E_i') \hookrightarrow C\Big(J; \sum_{i=1}^m (F_i',E_i')_{1-\xi_i,\frac{1}{1-\xi_i}}\Big).$$

Accordingly,

$$(u,v) \mapsto \big[t \mapsto \langle u(t), v(t) \rangle_\xi - \langle u(T_0), v(T_0) \rangle_\xi\big] \tag{A.3}$$

is continuous as a mapping from $\mathbb{W}^{1,\tau}_\varsigma(E,F) \times \mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$ to $C(\overline{J})$. Clearly,

$$(u,v) \mapsto \langle \partial_t u(t), v(t) \rangle_{F,F'} + \langle u(t), \partial_t v(t) \rangle_{E,E'}$$

maps $\mathbb{W}^{1,\tau}_\varsigma(E,F) \times \mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$ continuously into $L^1(J)$, hence

$$(u,v) \mapsto \big[t \mapsto \int_{T_0}^t \big\langle \partial_s u(s), v(s) \big\rangle_{F,F'} + \big\langle u(s), \partial_s v(s) \big\rangle_{E,E'} \, \mathrm{d}s\big] \tag{A.4}$$

is also a continuous mapping from $\mathbb{W}^{1,\tau}_\varsigma(E,F) \times \mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$ to $C(\overline{J})$.

Due to the *dense* embeddings $E_i \hookrightarrow (E_i,F_i)_{\zeta,p} \hookrightarrow F_i$ and $F_i' \hookrightarrow (F_i',E_i')_{\zeta,p} \hookrightarrow E'$ for all $\zeta \in (0,1)$ and all $p \in [1,\infty]$, the dual pairing $\langle u(t), v(t) \rangle_\xi$ coincides with $\langle u(t), v(t) \rangle_{E,F'}$ if $u(t) \in E$ and $v(t) \in F'$ ([2, Prop. V.1.4.8]). Thus, one calculates for $u \in C^1(\overline{J}) \otimes E$ and $v \in C^1(\overline{J}) \otimes F'$ that

$$\partial_t \Big[t \mapsto \big\langle u(t), v(t) \big\rangle_\xi\Big] = \big\langle \partial_t u(t), v(t) \big\rangle_{F,F'} + \big\langle u(t), \partial_t v(t) \big\rangle_{E,E'}$$

for all $t \in J$, and hence, by the fundamental theorem of calculus,

$$\big\langle u(t), v(t) \big\rangle_\xi - \big\langle u(T_0), v(T_0) \big\rangle_\xi = \int_{T_0}^t \big\langle \partial_s u(s), v(s) \big\rangle_{F,F'} + \big\langle u(s), \partial_s v(s) \big\rangle_{E,E'} \, \mathrm{d}s$$

for all $u \in C^1(\overline{J}) \otimes E$ and $v \in C^1(\overline{J}) \otimes F'$. But $C^1(\overline{J}) \otimes E$ and $C^1(\overline{J}) \otimes F'$ are dense in $C^1(J;E)$ and $C^1(J;F')$, which in turn are dense in $\mathbb{W}^{1,\tau}_\varsigma(E,F)$ and $\mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$ ([2, Thm. V.2.4.6]), and we had already noted that both sides of the preceding equation, seen as continuous functions in t, depend continuously w.r.t. to the sup-norm on $(u,v) \in \mathbb{W}^{1,\tau}_\varsigma(E,F) \times \mathbb{W}^{1,\varsigma'}_{\tau'}(F',E')$, cf. (A.3) and (A.4). From this, (A.2) follows. ∎

Appendix B. Non-Lipschitz projection counterexample

Lemma B.1. *Let* $\Lambda = (0,1)$ *and* $U_{\text{ad}} = \{u \in L^2(\Lambda)\colon u \geq 0 \text{ a.e. in } \Lambda\}$. *Then the projection* $P_{L^2(\Lambda)}$ *onto* U_{ad} *in* $L^2(\Lambda)$ *is not locally Lipschitz continuous on* $H_0^1(\Lambda)$.

Proof. Consider an equidistant decomposition of $\Lambda = (0,1)$ in $N \in \mathbb{N}$ subintervals of length $1/N$ with N even. Let $\varepsilon > 0$ be given. We consider the sawtooth functions

$$v_1(\mathrm{x}) := \begin{cases} \varepsilon N(\mathrm{x} - i/N) & \text{for } \mathrm{x} \in \big[i/N, (i+1)/N\big),\ i \in \{0, ..., N-2\} \text{ even}, \\ \varepsilon - \varepsilon N(\mathrm{x} - i/N) & \text{for } \mathrm{x} \in \big[i/N, (i+1)/N\big),\ i \in \{1, ..., N-1\} \text{ odd} \end{cases}$$

and

$$v_2(\mathrm{x}) := \begin{cases} 0 & \text{for } \mathrm{x} \in \big[0, 1/N\big), \\ -\varepsilon N(\mathrm{x} - i/N) & \text{for } \mathrm{x} \in \big[i/N, (i+1)/N\big),\ i \in \{1, ..., N-3\} \text{ odd}, \\ -\varepsilon + \varepsilon N(\mathrm{x} - i/N) & \text{for } \mathrm{x} \in \big[i/N, (i+1)/N\big),\ i \in \{2, ..., N-2\} \text{ even}, \\ 0 & \text{for } \mathrm{x} \in \big[(N-1)/N, 1\big). \end{cases}$$

The projection on U_{ad} in $L^2(\Lambda)$ is given by

$$P_{L^2(\Lambda)}(v)(\mathrm{x}) = \max\{v(\mathrm{x}), 0\} \quad \text{f.a.a. } \mathrm{x} \in \Lambda.$$

Note that $v_1 \in U_{\text{ad}}$, $v_2(\mathrm{x}) = v_1(\mathrm{x}) - \varepsilon$ for $\mathrm{x} \in [1/N, (N-1)/N]$, and $P_{L^2(\Lambda)}(v_2) \equiv 0$. Thus, we have

$$\big\|\nabla P_{L^2(\Lambda)}(v_1) - \nabla P_{L^2(\Lambda)}(v_2)\big\|^2_{L^2(\Lambda)} = \|\nabla v_1\|^2_{L^2(\Lambda)} = N \int_0^{1/N} |\varepsilon N\, \mathrm{x}|^2 \,\mathrm{dx} = \frac{1}{3}\,\varepsilon^2$$

and

$$\big\|\nabla v_1 - \nabla v_2\big\|^2_{L^2(\Lambda)} = 2 \int_0^{1/N} |\varepsilon N\, \mathrm{x}|^2 \,\mathrm{dx} = \frac{2}{N}\,\frac{1}{3}\,\varepsilon^2.$$

This implies that $P_{L^2(\Lambda)}$ is *even not locally Lipschitz continuous* in $H_0^1(\Lambda)$. To see this, assume the contrary that there are constants $\delta, L > 0$ such that

$$\big\|\nabla P_{L^2(\Lambda)}(v_1) - \nabla P_{L^2(\Lambda)}(v)\big\|_{L^2(\Lambda)} \leq L \,\big\|\nabla v_1 - \nabla v\big\|_{L^2(\Lambda)}$$

for all $v \in H_0^1(\Lambda)$ with $\|v - v_1\|_{H_0^1(\Lambda)} < \delta$. Then we choose $N > L/2$ and $\varepsilon < \sqrt{(2\delta)/(3N)}$ and obtain $\|v_2 - v_1\|_{H_0^1(\Lambda)} < \delta$, but

$$\big\|\nabla P_{L^2(\Lambda)}(v_1) - \nabla P_{L^2(\Lambda)}(v_2)\big\|_{L^2(\Lambda)} > L \,\big\|\nabla v_1 - \nabla v_2\big\|_{L^2(\Lambda)},$$

which is the desired contradiction. □

References

[1] Alber, Y. Metric and generalized projection operators in banach spaces: properties and applications. In *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. Kartsatos, Ed. Dekker, New York, 1996, pp. 15–50.

[2] Amann, H. *Linear and Quasilinear Parabolic Problems.* Birkhäuser Basel, 1995.

[3] Amann, H. Compact embeddings of vector-valued Sobolev and Besov spaces. *Glasnik Matematicki 35*, 55 (2000), 161–177.

[4] Aubin, J.-P. Un théoreme de compacité. *CR Acad. Sci. Paris 256*, 24 (1963), 5042–5044.

[5] Biegert, M. On traces of Sobolev functions on the boundary of extension domains. *Proceedings of the American Mathematical Society 137*, 12 (July 2009), 4169–4176.

[6] Bonnans, J. F., and Shapiro, A. *Pertubation Analysis of Optimization Problems.* Springer Series in Operations Research. Springer, 2000.

[7] Christof, C. *Sensitivity Analysis of Elliptic Variational Inequalities of the First and Second Kind.* PhD thesis, TU Dortmund, 2018.

[8] Christof, C., and Wachsmuth, G. Differential sensitivity analysis of variational inequalities with locally lipschitz continuous solution operators. Preprint, arXiv:1711.02720, 2017.

[9] Cioranescu, I. *Geometry of Banach spaces, duality mappings and nonlinear problems.* Kluwer Academic Publishers, 1990.

[10] Dautray, R., and Lions, J.-L. *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 5.* Springer-Verlag, Berlin, 1992. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

[11] Deville, R., Godefroy, G., and Zizler, V. *Smoothness and renormings in Banach spaces.* Longman Scientific & Technical, 1993.

[12] Diestel, J., and Uhl, J. *Vector measures.* American Mathematical Society, Providence, Rhode Island, 1977.

[13] Disser, K., Kaiser, H.-C., and Rehberg, J. Optimal Sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems. *SIAM J. Math. Anal. 47*, 3 (Jan. 2015), 1719–1746.

[14] Do, C. N. Generalized second-order derivatives of convex functions in reflexive banach spaces. *Trans. Amer. Math. Soc. 334*, 1 (1992), 281–301.

[15] Dubinskii, Y. A. Weak convergence for nonlinear elliptic and parabolic equations. *Mat. Sb. (N.S.) 67*, 4 (1965), 609–642. http://www.ams.org/mathscinet-getitem?mr=190546.

[16] Evans, L. C., and Gariepy, R. F. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[17] Gilbarg, D., and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order.* Springer Science + Business Media, 2001.

[18] Goldberg, H., and Tröltzsch, F. Second-order sufficient optimality conditions for a class of nonlinear parabolic boundary control problems. *SIAM J. Control Optim. 31*, 4 (July 1993), 1007–1025.

[19] Grisvard, P. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, Jan. 2011.

[20] Haller-Dintelmann, R., and Rehberg, J. Maximal parabolic regularity for divergence operators including mixed boundary conditions. *J. Differential Equations 247*, 5 (Sept. 2009), 1354–1396.

[21] Haller-Dintelmann, R., and Rehberg, J. Coercivity for elliptic operators and positivity of solutions on lipschitz domains. *Arch. Math. (Basel) 95*, 5 (Nov. 2010), 457–468.

[22] Haraux, A. How to differentiate the projection on a convex set in hilbert space. some applications to variational inequalities. *J. Math. Soc. Japan 29*, 4 (1977), 615–631.

[23] Hieber, M., and Rehberg, J. Quasilinear parabolic systems with mixed boundary conditions on nonsmooth domains. *SIAM Journal on Mathematical Analysis 40*, 1 (Jan. 2008), 292–305.

[24] Hömberg, D., Krumbiegel, K., and Rehberg, J. Optimal control of a parabolic equation with dynamic boundary condition. *Appl. Math. Optim. 67*, 1 (Aug. 2012), 3–31.

[25] Hömberg, D., Meyer, C., Rehberg, J., and Ring, W. Optimal control for the thermistor problem. *SIAM J. Control Optim. 48*, 5 (Jan. 2010), 3449–3481.

[26] Lang, S. *Real and Functional Analysis.* Springer New York, 1993.

[27] Leonard, I. E., and Sundaresan, K. Geometry of Lebesgue-Bochner function spaces–smoothness. *Trans. Amer. Math. Soc. 198* (1974), 229–229.

[28] Leonard, I. E., and Sundaresan, K. Smoothness and duality in $l^p(e, \mu)$. *J. Math. Anal. Appl. 46* (1974), 513–522.

[29] Li, J. The generalized projection operator on reflexive banach spaces and its applications. *J. Math. Anal. Appl. 306*, 1 (June 2005), 55–71.

[30] Lions, J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris, 1969.

[31] Maz'ya, V. *Sobolev Spaces.* Springer Berlin Heidelberg, 2011.

[32] Meinlschmidt, H., Meyer, C., and Rehberg, J. Optimal control of the thermistor problem in three spatial dimensions, part 1: Existence of optimal solutions. *SIAM J. Control Optim. 55*, 5 (2017), 2876–2904.

[33] Meinlschmidt, H., Meyer, C., and Rehberg, J. Optimal control of the thermistor problem in three spatial dimensions, part 2: Optimality conditions. *SIAM J. Control Optim. 55*, 4 (2017), 2368–2392.

[34] Mignot, F. Contrôle dans les inéquations variationelles elliptiques. *Journal of Functional Analysis 22*, 2 (1976), 130–185.

[35] Murat, F. Compacité par compensation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. 4* (1978), 489–507. French.

[36] Neitzel, I., and Tröltzsch, F. On convergence of regularization methods for nonlinear parabolic optimal control problems with control and state constraints. *Control Cybernet. 37*, 4 (2008), 1013–1043.

[37] Neitzel, I., and Vexler, B. A priori error estimates for space–time finite element discretization of semilinear parabolic optimal control problems. *Numer. Math. 120*, 2 (Sept. 2011), 345–386.

[38] Nirenberg, L. On elliptic partial differential equations. *Ann. Scuola. Norm. Sup. Pisa 13* (1959), 115–162.

[39] Prüss, J., and Schnaubelt, R. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl. 256*, 2 (Apr. 2001), 405–430.

[40] Raymond, J.-P., and Tröltzsch, F. Second order sufficient optimality conditions for nonlinear parabolic control problems with state constraints. *Discrete Continuous Dynamical Systems 6* (2006), 431–450.

[41] Schirotzek, W. *Nonsmooth Analysis.* Springer-Verlag Berlin Heidelberg, 2007.

[42] Schuster, T., Kaltenbacher, B., Hofmann, B., and Kazimierski, K. *Regularization in Banach Spaces*. De Gruyter Berlin, Boston, 2012.

[43] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4) 146*, 1 (Dec. 1986), 65–96.

[44] Tartar, L. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symp.*, R. J. Knops, Ed., vol. 4 of *Nonlinear analysis and mechanics, Heriot-Watt Symposium*. Pitman Press, London, 1979.

[45] ter Elst, A., Meyries, M., and Rehberg, J. Parabolic equations with dynamical boundary conditions and source terms on interfaces. *Ann. Mat. Pura Appl. 193*, 5 (2014), 1295–1318.

[46] Triebel, H. *Interpolation theory, function spaces, differential operators*, vol. 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[47] Ulbrich, M. *Semismooth Newton Methods for Variational Inequalities and Constrained Optimization Problems in Function Spaces*. Society for Industrial & Applied Mathematics (SIAM), Jan. 2011.

[48] Wachsmuth, G. A guided tour of polyhedric sets: Basic properties, new results on intersections and applications. *Journal of Convex Analysis 26*, 1 (2019), 153–188.

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